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Pearson Edexcel
Qualifications

MATHEMATICS

eBook included

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 1

STUDENT BOOK

PEARSON EDEXCEL INTERNATIONAL A LEVEL

PURE MATHEMATICS 1

Student Book

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

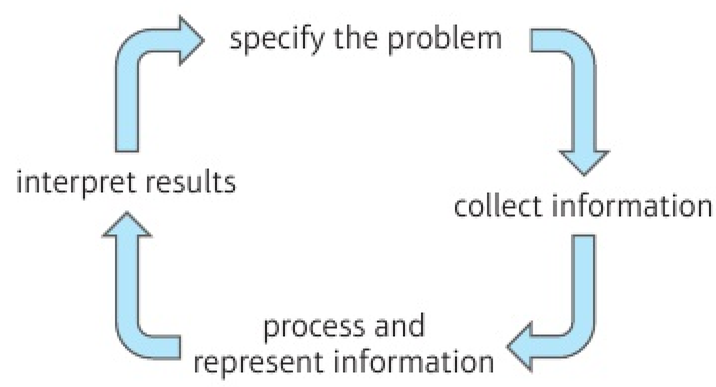
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

The Mathematical Problem-Solving Cycle



3. Transferable skills

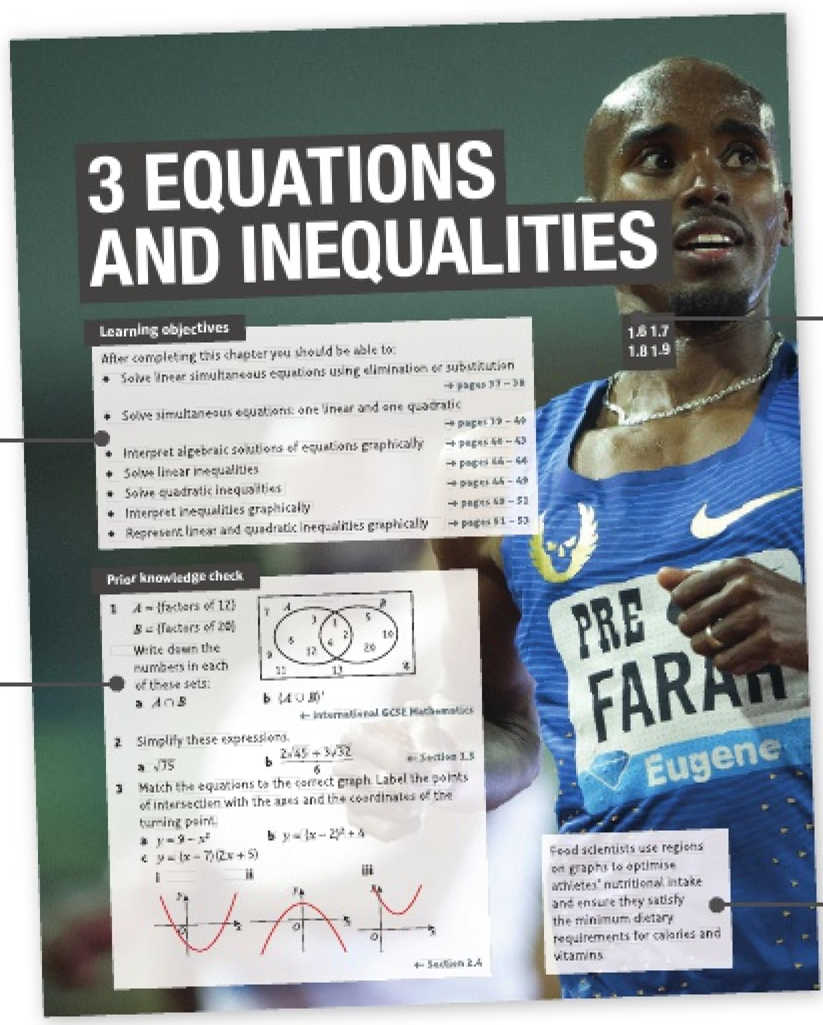
- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book

Each chapter starts with a list of *Learning objectives*

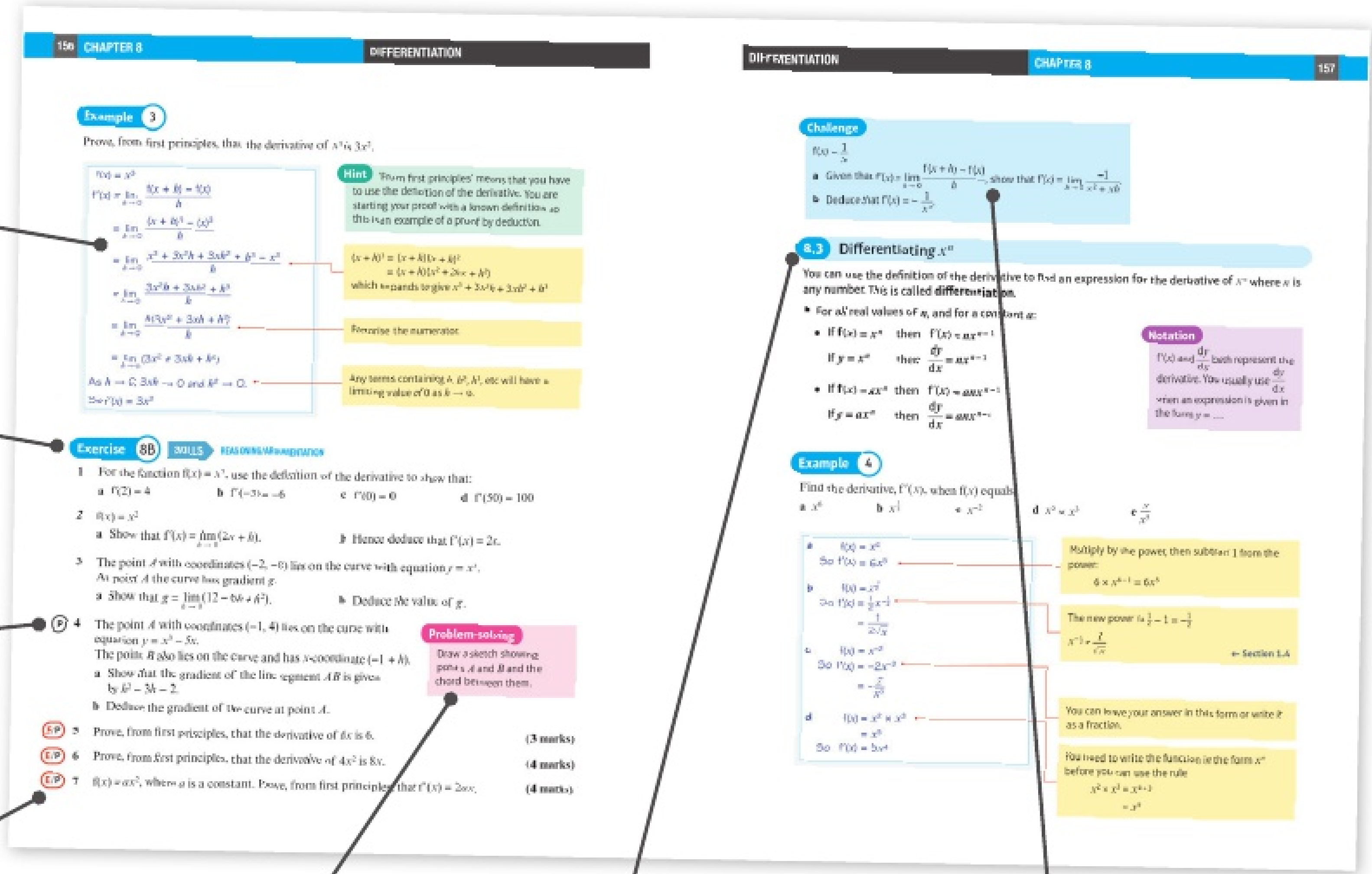
The *Prior knowledge check* helps make sure you are ready to start the chapter

Glossary terms will be identified by bold blue text, on their first appearance



Each chapter is mapped to the specification content for easy reference

The real world applications of the maths you are about to learn are highlighted at the start of the chapter



Step-by-step worked examples focus on the key types of questions you'll need to tackle

Transferable skills are signposted where they naturally occur in the exercises and examples

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

Each section begins with an explanation and key learning points

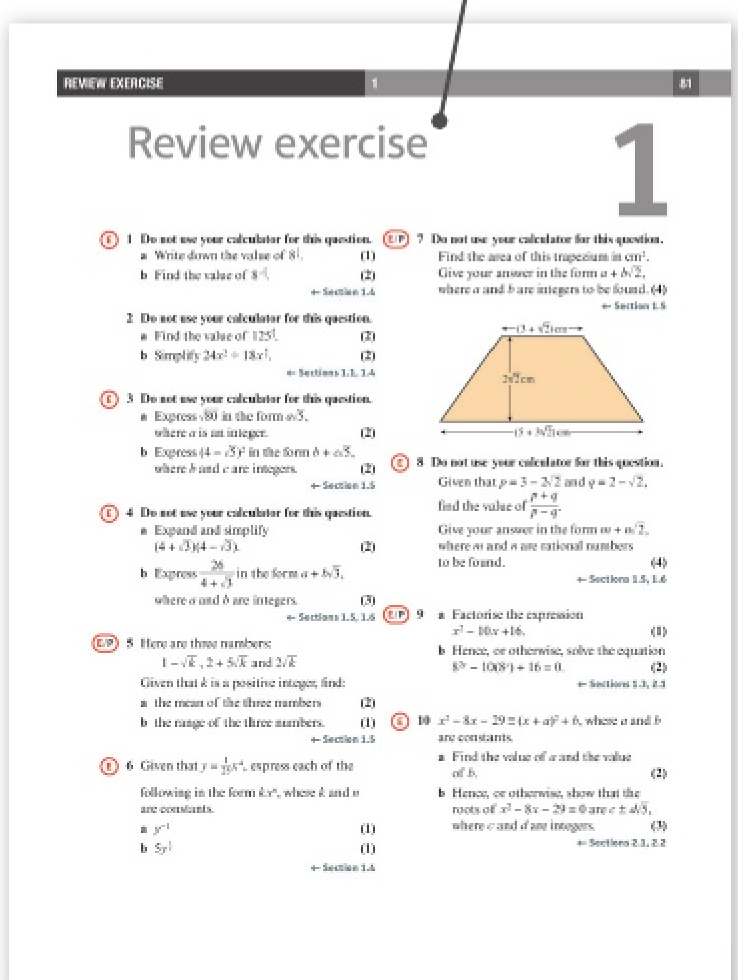
Challenge boxes give you a chance to tackle some more difficult questions

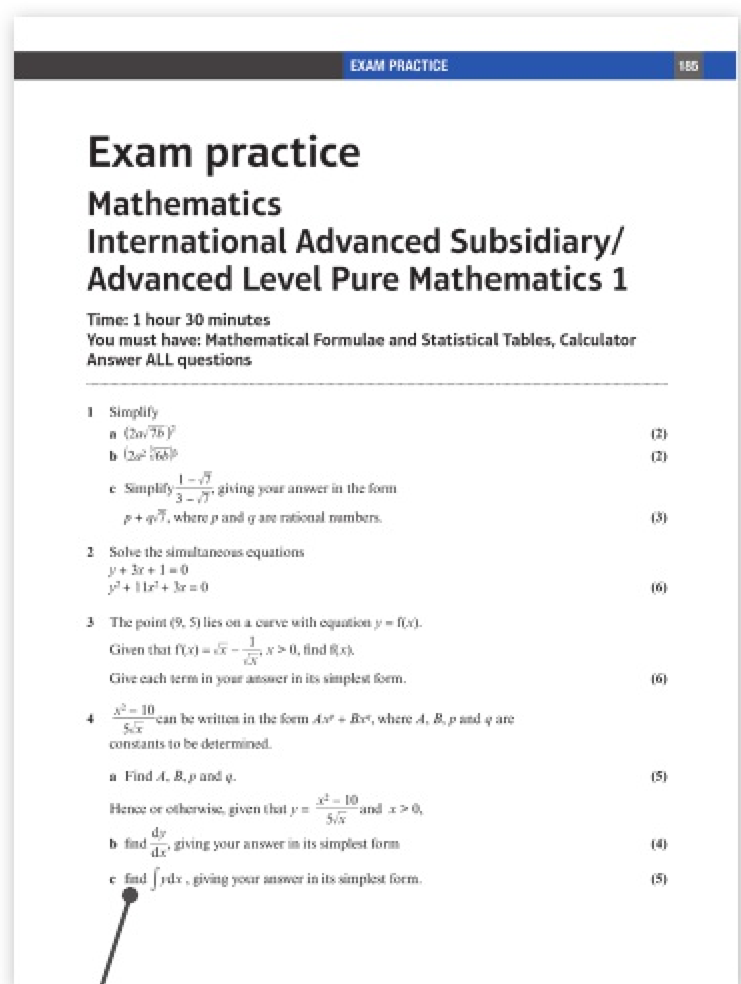
Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Pure Mathematics 1 (P1) is a **compulsory** unit in the following qualifications:

International Advanced Subsidiary in Mathematics

International Advanced Subsidiary in Pure Mathematics

International Advanced Level in Mathematics

International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
P1: Pure Mathematics 1 Paper code WMA11/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January, June and October First assessment January 2019

IAS – International Advanced Subsidiary, IAL – International Advanced A Level

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

P1	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	30–35	25–30	5–15	5–10	1–5
%	40-$46\frac{2}{3}$	$33\frac{1}{3}$–40	$6\frac{2}{3}$–20	$6\frac{2}{3}$–$13\frac{1}{3}$	$1\frac{1}{3}$–$6\frac{2}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements outlined below.

Students are expected to have available a calculator with at least the following keys: +, −, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, x!, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides worked solutions for questions in the book. Download all the solutions as a PDF or quickly find the solution you need online.

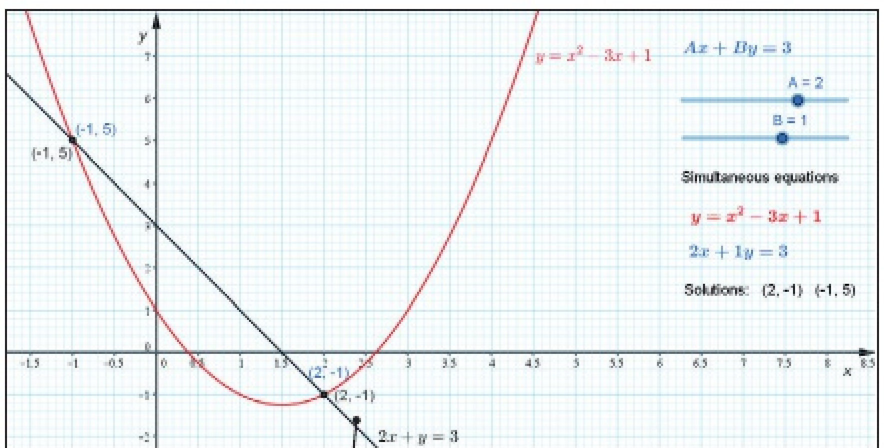
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

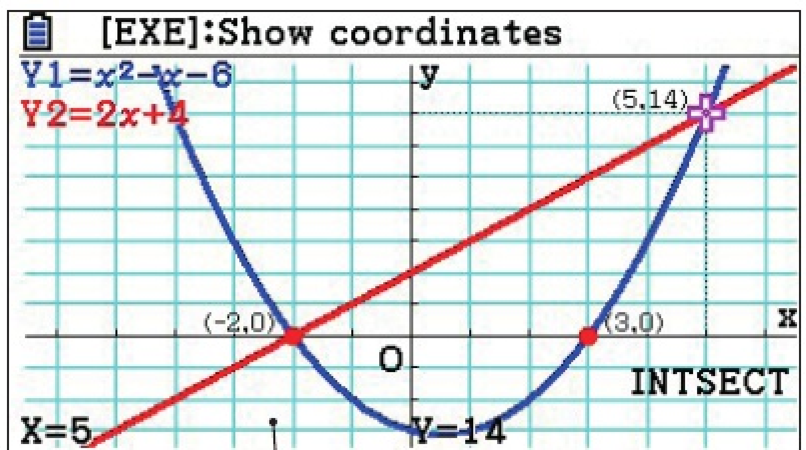
GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

CASIO

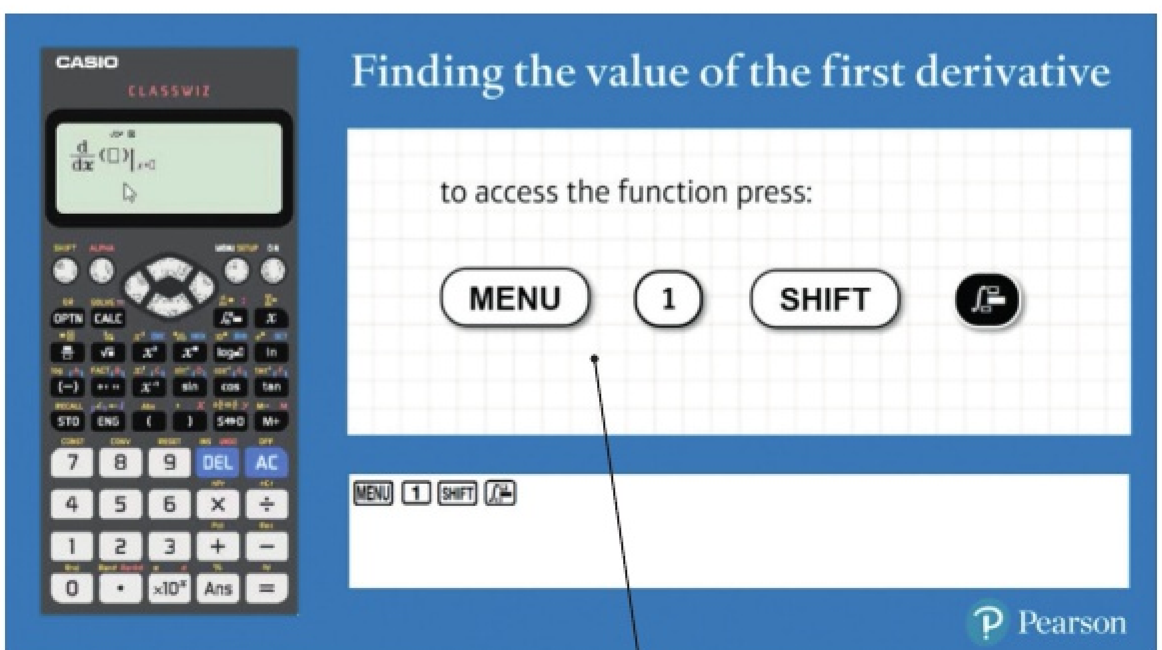
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.

1 ALGEBRAIC EXPRESSIONS

1.1
1.2
1.10

Learning objectives

After completing this chapter you should be able to:

- Multiply and divide integer powers → pages 2–4
- Expand a single term over brackets and collect like terms → pages 2–4
- Expand the product of two or three expressions → pages 4–6
- Factorise linear, quadratic and simple cubic expressions → pages 6–9
- Know and use the laws of indices → pages 9–11
- Simplify and use the rules of surds → pages 12–13
- Rationalise denominators → pages 13–15

Prior knowledge check

1 Simplify:

a $4m^2n + 5mn^2 - 2m^2n + mn^2 - 3mn^2$

b $3x^2 - 5x + 2 + 3x^2 - 7x - 12$

← International GCSE Mathematics

2 Write as a single power of 2:

a $2^5 \times 2^3$ **b** $2^6 \div 2^2$

c $(2^3)^2$ ← International GCSE Mathematics

3 Expand:

a $3(x + 4)$ **b** $5(2 - 3x)$

c $6(2x - 5y)$ ← International GCSE Mathematics

4 Write down the highest common factor of:

a 24 and 16 **b** $6x$ and $8x^2$

c $4xy^2$ and $3xy$ ← International GCSE Mathematics

5 Simplify:

a $\frac{10x}{5}$ **b** $\frac{20x}{2}$ **c** $\frac{40x}{24}$

← International GCSE Mathematics

Computer scientists use indices to describe very large numbers. A quantum computer with 1000 qubits (quantum bits) can consider 2^{1000} values simultaneously. This is greater than the number of particles in the observable universe.

1.1 Index laws

- You can use the laws of **indices** to **simplify** powers of the same **base**.
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

Notation

x^5 — the 5: This is the **index**, **power** or **exponent**. The x: This is the base.

Example 1

Simplify these **expressions**:

a $x^2 \times x^5$ **b** $2r^2 \times 3r^3$ **c** $\frac{b^7}{b^4}$ **d** $6x^5 \div 3x^3$ **e** $(a^3)^2 \times 2a^2$ **f** $(3x^2)^3 \div x^4$

a $x^2 \times x^5 = x^{2+5} = x^7$

Use the rule $a^m \times a^n = a^{m+n}$ to simplify the index.

b $2r^2 \times 3r^3 = 2 \times 3 \times r^2 \times r^3$
$= 6 \times r^{2+3} = 6r^5$

Rewrite the expression with the numbers together and the r **terms** together.
$2 \times 3 = 6$
$r^2 \times r^3 = r^{2+3}$

c $\frac{b^7}{b^4} = b^{7-4} = b^3$

Use the rule $a^m \div a^n = a^{m-n}$ to simplify the index.

d $6x^5 \div 3x^3 = \frac{6}{3} \times \frac{x^5}{x^3}$
$= 2 \times x^2 = 2x^2$

$x^5 \div x^3 = x^{5-3} = x^2$

e $(a^3)^2 \times 2a^2 = a^6 \times 2a^2$
$= 2 \times a^6 \times a^2 = 2a^8$

Use the rule $(a^m)^n = a^{mn}$ to simplify the index.

$a^6 \times a^2 = a^{6+2} = a^8$

f $\frac{(3x^2)^3}{x^4} = 3^3 \times \frac{(x^2)^3}{x^4}$
$= 27 \times \frac{x^6}{x^4} = 27x^2$

Use the rule $(ab)^n = a^n b^n$ to simplify the **numerator**.
$(x^2)^3 = x^{2\times 3} = x^6$

$\frac{x^6}{x^4} = x^{6-4} = x^2$

Example 2

Expand these expressions and simplify if possible:

a $-3x(7x - 4)$ **b** $y^2(3 - 2y^3)$

c $4x(3x - 2x^2 + 5x^3)$ **d** $2x(5x + 3) - 5(2x + 3)$

Watch out A minus sign outside brackets changes the sign of every term inside the brackets.

a $-3x(7x-4) = -21x^2 + 12x$

b $y^2(3-2y^3) = 3y^2 - 2y^5$

c $4x(3x - 2x^2 + 5x^3)$

$= 12x^2 - 8x^3 + 20x^4$

d $2x(5x+3) - 5(2x+3)$

$= 10x^2 + 6x - 10x - 15$

$= 10x^2 - 4x - 15$

$-3x \times 7x = -21x^{1+1} = -21x^2$

$-3x \times (-4) = +12x$

$y^2 \times (-2y^3) = -2y^{2+3} = -2y^5$

Remember: a minus sign outside the brackets changes the signs within the brackets.

Simplify $6x - 10x$ to give $-4x$.

Example 3

Simplify these expressions:

a $\dfrac{x^7 + x^4}{x^3}$ b $\dfrac{3x^2 - 6x^5}{2x}$ c $\dfrac{20x^7 + 15x^3}{5x^2}$

a $\dfrac{x^7 + x^4}{x^3} = \dfrac{x^7}{x^3} + \dfrac{x^4}{x^3}$

$= x^{7-3} + x^{4-3} = x^4 + x$

b $\dfrac{3x^2 - 6x^5}{2x} = \dfrac{3x^2}{2x} - \dfrac{6x^5}{2x}$

$= \dfrac{3}{2}x^{2-1} - 3x^{5-1} = \dfrac{3x}{2} - 3x^4$

c $\dfrac{20x^7 + 15x^3}{5x^2} = \dfrac{20x^7}{5x^2} + \dfrac{15x^3}{5x^2}$

$= 4x^{7-2} + 3x^{3-2} = 4x^5 + 3x$

Divide each term of the numerator by x^3.

x^1 is the same as x.

Divide each term of the numerator by $2x$.

Simplify each fraction:

$\dfrac{3x^2}{2x} = \dfrac{3}{2} \times \dfrac{x^2}{x} = \dfrac{3}{2} \times x^{2-1}$

$-\dfrac{6x^5}{2x} = -\dfrac{6}{2} \times \dfrac{x^5}{x} = -3 \times x^{5-1}$

Divide each term of the numerator by $5x^2$.

Exercise 1A SKILLS INTERPRETATION

1 Simplify these expressions:

a $x^3 \times x^4$ b $2x^3 \times 3x^2$ c $\dfrac{k^3}{k^2}$

d $\dfrac{4p^3}{2p}$ e $\dfrac{3x^3}{3x^2}$ f $(y^2)^5$

g $10x^5 \div 2x^3$ h $(p^3)^2 \div p^4$ i $(2a^3)^2 \div 2a^3$

j $8p^4 \div 4p^3$ k $2a^4 \times 3a^5$ l $\dfrac{21a^3b^7}{7ab^4}$

m $9x^2 \times 3(x^2)^3$ n $3x^3 \times 2x^2 \times 4x^6$ o $7a^4 \times (3a^4)^2$

p $(4y^3)^3 \div 2y^3$ q $2a^3 \div 3a^2 \times 6a^5$ r $3a^4 \times 2a^5 \times a^3$

2 Expand and simplify if possible:

a $9(x-2)$ **b** $x(x+9)$ **c** $-3y(4-3y)$
d $x(y+5)$ **e** $-x(3x+5)$ **f** $-5x(4x+1)$
g $(4x+5)x$ **h** $-3y(5-2y^2)$ **i** $-2x(5x-4)$
j $(3x-5)x^2$ **k** $3(x+2)+(x-7)$ **l** $5x-6-(3x-2)$
m $4(c+3d^2)-3(2c+d^2)$ **n** $(r^2+3t^2+9)-(2r^2+3t^2-4)$
o $x(3x^2-2x+5)$ **p** $7y^2(2-5y+3y^2)$ **q** $-2y^2(5-7y+3y^2)$
r $7(x-2)+3(x+4)-6(x-2)$ **s** $5x-3(4-2x)+6$
t $3x^2-x(3-4x)+7$ **u** $4x(x+3)-2x(3x-7)$ **v** $3x^2(2x+1)-5x^2(3x-4)$

3 Simplify these fractions:

a $\dfrac{6x^4+10x^6}{2x}$ **b** $\dfrac{3x^5-x^7}{x}$ **c** $\dfrac{2x^4-4x^2}{4x}$
d $\dfrac{8x^3+5x}{2x}$ **e** $\dfrac{7x^7+5x^2}{5x}$ **f** $\dfrac{9x^5-5x^3}{3x}$

1.2 Expanding brackets

To find the **product** of two expressions you **multiply** each term in one expression by each term in the other expression.

Multiplying each of the 2 terms in the first expression by each of the 3 terms in the second expression gives 2 × 3 = 6 terms.

$x\times$ / $5\times$

$$(x+5)(4x-2y+3) = x(4x-2y+3)+5(4x-2y+3)$$
$$= 4x^2-2xy+3x+20x-10y+15$$
$$= 4x^2-2xy+23x-10y+15$$

Simplify your answer by **collecting like terms**.

Example 4 SKILLS INTERPRETATION

Expand these expressions and simplify if possible:

a $(x+5)(x+2)$ **b** $(x-2y)(x^2+1)$ **c** $(x-y)^2$ **d** $(x+y)(3x-2y-4)$

a $(x+5)(x+2)$ — Multiply x by $(x+2)$ and then multiply 5 by $(x+2)$.
$= x^2+2x+5x+10$ — Simplify your answer by collecting like terms.
$= x^2+7x+10$

b $(x-2y)(x^2+1)$ — $-2y\times x^2=-2x^2y$
$= x^3+x-2x^2y-2y$ — There are no like terms to collect.

c $(x-y)^2$
$= (x-y)(x-y)$
$= x^2 - xy - xy + y^2$
$= x^2 - 2xy + y^2$

$(x-y)^2$ means $(x-y)$ multiplied by itself.

$-xy - xy = -2xy$

d $(x+y)(3x-2y-4)$
$= x(3x-2y-4) + y(3x-2y-4)$
$= 3x^2 - 2xy - 4x + 3xy - 2y^2 - 4y$
$= 3x^2 + xy - 4x - 2y^2 - 4y$

Multiply x by $(3x-2y-4)$ and then multiply y by $(3x-2y-4)$.

Example 5

Expand these expressions and simplify if possible:

a $x(2x+3)(x-7)$ **b** $x(5x-3y)(2x-y+4)$ **c** $(x-4)(x+3)(x+1)$

a $x(2x+3)(x-7)$
$= (2x^2+3x)(x-7)$
$= 2x^3 - 14x^2 + 3x^2 - 21x$
$= 2x^3 - 11x^2 - 21x$

Start by expanding one pair of brackets: $x(2x+3) = 2x^2 + 3x$

You could also have expanded the second pair of brackets first: $(2x+3)(x-7) = 2x^2 - 11x - 21$ Then multiply by x.

b $x(5x-3y)(2x-y+4)$
$= (5x^2-3xy)(2x-y+4)$
$= 5x^2(2x-y+4) - 3xy(2x-y+4)$
$= 10x^3 - 5x^2y + 20x^2 - 6x^2y + 3xy^2 - 12xy$
$= 10x^3 - 11x^2y + 20x^2 + 3xy^2 - 12xy$

Be careful with minus signs. You need to change every sign in the second pair of brackets when you multiply it out.

c $(x-4)(x+3)(x+1)$
$= (x^2-x-12)(x+1)$
$= x^2(x+1) - x(x+1) - 12(x+1)$
$= x^3 + x^2 - x^2 - x - 12x - 12$
$= x^3 - 13x - 12$

Choose one pair of brackets to expand first, for example:
$(x-4)(x+3) = x^2 + 3x - 4x - 12$
$= x^2 - x - 12$

You multiplied together three **linear** terms, so the final answer contains an x^3 term.

Exercise 1B SKILLS INTERPRETATION

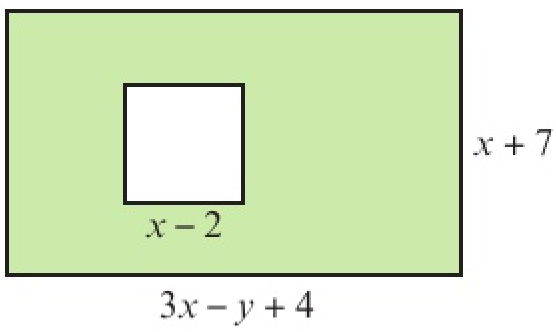
1 Expand and simplify if possible:

a $(x+4)(x+7)$	**b** $(x-3)(x+2)$	**c** $(x-2)^2$
d $(x-y)(2x+3)$	**e** $(x+3y)(4x-y)$	**f** $(2x-4y)(3x+y)$
g $(2x-3)(x-4)$	**h** $(3x+2y)^2$	**i** $(2x+8y)(2x+3)$
j $(x+5)(2x+3y-5)$	**k** $(x-1)(3x-4y-5)$	**l** $(x-4y)(2x+y+5)$
m $(x+2y-1)(x+3)$	**n** $(2x+2y+3)(x+6)$	**o** $(4-y)(4y-x+3)$
p $(4y+5)(3x-y+2)$	**q** $(5y-2x+3)(x-4)$	**r** $(4y-x-2)(5-y)$

2 Expand and simplify if possible:

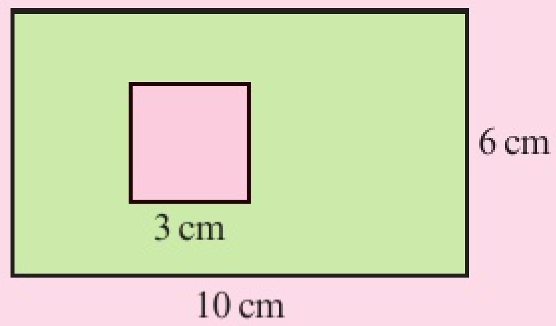
a $5(x+1)(x-4)$ **b** $7(x-2)(2x+5)$ **c** $3(x-3)(x-3)$
d $x(x-y)(x+y)$ **e** $x(2x+y)(3x+4)$ **f** $y(x-5)(x+1)$
g $y(3x-2y)(4x+2)$ **h** $y(7-x)(2x-5)$ **i** $x(2x+y)(5x-2)$
j $x(x+2)(x+3y-4)$ **k** $y(2x+y-1)(x+5)$ **l** $y(3x+2y-3)(2x+1)$
m $x(2x+3)(x+y-5)$ **n** $2x(3x-1)(4x-y-3)$ **o** $3x(x-2y)(2x+3y+5)$
p $(x+3)(x+2)(x+1)$ **q** $(x+2)(x-4)(x+3)$ **r** $(x+3)(x-1)(x-5)$
s $(x-5)(x-4)(x-3)$ **t** $(2x+1)(x-2)(x+1)$ **u** $(2x+3)(3x-1)(x+2)$
v $(3x-2)(2x+1)(3x-2)$ **w** $(x+y)(x-y)(x-1)$ **x** $(2x-3y)^3$

(P) **3** The diagram shows a rectangle with a square cut out.
The rectangle has length $3x - y + 4$ and width $x + 7$.
The square has side length $x - 2$.
Find an expanded and simplified expression for the area **shaded** green.



Problem-solving

Use the same strategy as you would use if the lengths were given as numbers:



(P) **4** A **cuboid** has **dimensions** $(x+2)$ cm, $(2x-1)$ cm and $(2x+3)$ cm.
Show that the volume of the cuboid is $(4x^3 + 12x^2 + 5x - 6)\text{ cm}^3$.

(E/P) **5** **Given that** $(2x+5y)(3x-y)(2x+y) = ax^3 + bx^2y + cxy^2 + dy^3$,
where a, b, c and d are **constants**, find the values of a, b, c and d. **(2 marks)**

Challenge

Expand and simplify $(x+y)^4$.

1.3 Factorising

You can write expressions as a **product** of their **factors**.

- Factorising is the opposite of expanding brackets.

Expanding brackets →

$4x(2x+y) = 8x^2 + 4xy$

$(x+5)^3 = x^3 + 15x^2 + 75x + 125$

$(x+2y)(x-5y) = x^2 - 3xy - 10y^2$

← Factorising

Example 6 SKILLS ANALYSIS

Factorise these expressions completely:

a $3x + 9$ **b** $x^2 - 5x$ **c** $8x^2 + 20x$ **d** $9x^2y + 15xy^2$ **e** $3x^2 - 9xy$

a $3x + 9 = 3(x + 3)$ — 3 is a **common factor** of $3x$ and 9.

b $x^2 - 5x = x(x - 5)$ — x is a common factor of x^2 and $-5x$.

c $8x^2 + 20x = 4x(2x + 5)$ — 4 and x are common factors of $8x^2$ and $20x$, so take $4x$ outside the brackets.

d $9x^2y + 15xy^2 = 3xy(3x + 5y)$ — 3, x and y are common factors of $9x^2y$ and $15xy^2$, so take $3xy$ outside the brackets.

e $3x^2 - 9xy = 3x(x - 3y)$ — x and $-3y$ have no common factors so this expression is completely factorised.

■ A **quadratic** expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

Notation **Real** numbers are all the positive and negative numbers, or zero, including fractions and **surds**.

To factorise a quadratic expression:

- Find two factors of ac that add up to b — For the expression $2x^2 + 5x - 3$, $ac = -6 = -1 \times 6$ and $-1 + 6 = 5 = b$.
- Rewrite the b term as a sum of these two factors — $2x^2 - x + 6x - 3$
- Factorise each pair of terms — $= x(2x - 1) + 3(2x - 1)$
- Take out the common factor — $= (2x - 1)(x + 3)$

■ $x^2 - y^2 = (x + y)(x - y)$

Notation An expression in the form $x^2 - y^2$ is called the **difference** of two squares.

Example 7

Factorise:

a $x^2 - 5x - 6$ **b** $x^2 + 6x + 8$ **c** $6x^2 - 11x - 10$ **d** $x^2 - 25$ **e** $4x^2 - 9y^2$

a $x^2 - 5x - 6$

$ac = -6$ and $b = -5$

So $x^2 - 5x - 6 = x^2 + x - 6x - 6$

$= x(x + 1) - 6(x + 1)$

$= (x + 1)(x - 6)$

Here $a = 1$, $b = -5$ and $c = -6$.

(1) Find the two factors of $ac = -6$ which add to give $b = -5$. $-6 + 1 = -5$

(2) Rewrite the b term using these two factors.

(3) Factorise the first two terms and the last two terms.

(4) $x + 1$ is a factor of both terms, so take that outside the brackets. This is now completely factorised.

b $x^2 + 6x + 8$
$= x^2 + 2x + 4x + 8$ — $ac = 8$ and $2 + 4 = 6 = b$.
$= x(x + 2) + 4(x + 2)$ — Factorise.
$= (x + 2)(x + 4)$

c $6x^2 - 11x - 10$
$= 6x^2 - 15x + 4x - 10$ — $ac = -60$ and $4 - 15 = -11 = b$.
$= 3x(2x - 5) + 2(2x - 5)$ — Factorise.
$= (2x - 5)(3x + 2)$

d $x^2 - 25$ — This is the difference of two squares as the two terms are x^2 and 5^2.
$= x^2 - 5^2$
$= (x + 5)(x - 5)$ — The two x terms, $5x$ and $-5x$, **cancel** each other out.

e $4x^2 - 9y^2$ — This is the same as $(2x)^2 - (3y)^2$.
$= 2^2x^2 - 3^2y^2$
$= (2x + 3y)(2x - 3y)$

Example 8

Factorise completely:

a $x^3 - 2x^2$ **b** $x^3 - 25x$ **c** $x^3 + 3x^2 - 10x$

a $x^3 - 2x^2 = x^2(x - 2)$ — You can't factorise this any further.

b $x^3 - 25x = x(x^2 - 25)$ — x is a common factor of x^3 and $-25x$, so take x outside the brackets. $x^2 - 25$ is the difference of two squares.
$= x(x^2 - 5^2)$
$= x(x + 5)(x - 5)$

c $x^3 + 3x^2 - 10x = x(x^2 + 3x - 10)$ — Write the expression as a product of x and a quadratic factor.
$= x(x + 5)(x - 2)$ — Factorise the quadratic to get three linear factors.

Exercise 1C

SKILLS PROBLEM-SOLVING

1 Factorise these expressions completely:

a $4x + 8$ **b** $6x - 24$ **c** $20x + 15$
d $2x^2 + 4$ **e** $4x^2 + 20$ **f** $6x^2 - 18x$
g $x^2 - 7x$ **h** $2x^2 + 4x$ **i** $3x^2 - x$
j $6x^2 - 2x$ **k** $10y^2 - 5y$ **l** $35x^2 - 28x$
m $x^2 + 2x$ **n** $3y^2 + 2y$ **o** $4x^2 + 12x$
p $5y^2 - 20y$ **q** $9xy^2 + 12x^2y$ **r** $6ab - 2ab^2$
s $5x^2 - 25xy$ **t** $12x^2y + 8xy^2$ **u** $15y - 20yz^2$
v $12x^2 - 30$ **w** $xy^2 - x^2y$ **x** $12y^2 - 4yx$

2 Factorise:

a $x^2 + 4x$ **b** $2x^2 + 6x$ **c** $x^2 + 11x + 24$
d $x^2 + 8x + 12$ **e** $x^2 + 3x - 40$ **f** $x^2 - 8x + 12$
g $x^2 + 5x + 6$ **h** $x^2 - 2x - 24$ **i** $x^2 - 3x - 10$
j $x^2 + x - 20$ **k** $2x^2 + 5x + 2$ **l** $3x^2 + 10x - 8$
m $5x^2 - 16x + 3$ **n** $6x^2 - 8x - 8$
o $2x^2 + 7x - 15$ **p** $2x^4 + 14x^2 + 24$
q $x^2 - 4$ **r** $x^2 - 49$
s $4x^2 - 25$ **t** $9x^2 - 25y^2$ **u** $36x^2 - 4$
v $2x^2 - 50$ **w** $6x^2 - 10x + 4$ **x** $15x^2 + 42x - 9$

Hint For part **n**, take 2 out as a common factor first. For part **p**, let $y = x^2$.

3 Factorise completely:

a $x^3 + 2x$ **b** $x^3 - x^2 + x$ **c** $x^3 - 5x$
d $x^3 - 9x$ **e** $x^3 - x^2 - 12x$ **f** $x^3 + 11x^2 + 30x$
g $x^3 - 7x^2 + 6x$ **h** $x^3 - 64x$ **i** $2x^3 - 5x^2 - 3x$
j $2x^3 + 13x^2 + 15x$ **k** $x^3 - 4x$ **l** $3x^3 + 27x^2 + 60x$

E/P **4** Factorise completely $x^4 - y^4$. **(2 marks)**

Problem-solving
Watch out for terms that can be written as a function of a function, for example: $x^4 = (x^2)^2$.

E **5** Factorise completely $6x^3 + 7x^2 - 5x$. **(2 marks)**

Challenge
Write $4x^4 - 13x^2 + 9$ as the product of four linear factors.

1.4 Negative and fractional indices

Indices can be negative numbers or fractions.

$x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x,$

similarly $\underbrace{x^{\frac{1}{n}} \times x^{\frac{1}{n}} \times \ldots \times x^{\frac{1}{n}}}_{n \text{ terms}} = x^{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}} = x^1 = x$

Notation **Rational** numbers are those that can be written as $\frac{a}{b}$ where a and b are **integers**, and $b \neq 0$.

- You can use the laws of indices with any rational power.
  - $a^{\frac{1}{m}} = \sqrt[m]{a}$
  - $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
  - $a^{-m} = \frac{1}{a^m}$
  - $a^0 = 1$

Notation $a^{\frac{1}{2}} = \sqrt{a}$ is the positive **square root** of a. For example: $9^{\frac{1}{2}} = \sqrt{9} = 3$, but $9^{\frac{1}{2}} \neq -3$.

Example 9

Simplify:

a $\frac{x^3}{x^{-3}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$ **c** $(x^3)^{\frac{2}{3}}$ **d** $2x^{1.5} \div 4x^{-0.25}$ **e** $\sqrt[3]{125x^6}$ **f** $\frac{2x^2 - x}{x^5}$

a $\frac{x^3}{x^{-3}} = x^{3-(-3)} = x^6$

Use the rule $a^m \div a^n = a^{m-n}$.

b $x^{\frac{1}{2}} \times x^{\frac{3}{2}} = x^{\frac{1}{2}+\frac{3}{2}} = x^2$

This could also be written as $\sqrt{x}$.
Use the rule $a^m \times a^n = a^{m+n}$.

c $(x^3)^{\frac{2}{3}} = x^{3 \times \frac{2}{3}} = x^2$

Use the rule $(a^m)^n = a^{mn}$.

d $2x^{1.5} \div 4x^{-0.25} = \frac{1}{2}x^{1.5-(-0.25)} = \frac{1}{2}x^{1.75}$

Use the rule $a^m \div a^n = a^{m-n}$.
$1.5 - (-0.25) = 1.75$

e $\sqrt[3]{125x^6} = (125x^6)^{\frac{1}{3}}$

$= (125)^{\frac{1}{3}}(x^6)^{\frac{1}{3}} = \sqrt[3]{125}(x^{6 \times \frac{1}{3}}) = 5x^2$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$.

f $\frac{2x^2 - x}{x^5} = \frac{2x^2}{x^5} - \frac{x}{x^5}$

Divide each term of the numerator by x^5.

$= 2 \times x^{2-5} - x^{1-5} = 2x^{-3} - x^{-4}$

$= \frac{2}{x^3} - \frac{1}{x^4}$

Using $a^{-m} = \frac{1}{a^m}$

Example 10 SKILLS INTERPRETATION

Evaluate:

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $25^{-\frac{3}{2}}$

a $9^{\frac{1}{2}} = \sqrt{9} = 3$

Using $a^{\frac{1}{m}} = \sqrt[m]{a}$. Thus, $9^{\frac{1}{2}} = \sqrt{9}$.

b $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$

This means the cube root of 64.

c $49^{\frac{3}{2}} = (\sqrt{49})^3$

$= 7^3 = 343$

Using $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.
This means the square root of 49, cubed.

d $25^{-\frac{3}{2}} = \frac{1}{25^{\frac{3}{2}}} = \frac{1}{(\sqrt{25})^3}$

$= \frac{1}{5^3} = \frac{1}{125}$

Using $a^{-m} = \frac{1}{a^m}$

Online Use your calculator to enter negative and **fractional** powers.

Example 11

Given that $y = \frac{1}{16}x^2$, **express** each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{2}}$ **b** $4y^{-1}$

a $y^{\frac{1}{2}} = \left(\frac{1}{16}x^2\right)^{\frac{1}{2}}$

$= \frac{1}{\sqrt{16}}x^{2 \times \frac{1}{2}} = \frac{x}{4}$

b $4y^{-1} = 4\left(\frac{1}{16}x^2\right)^{-1}$

$= 4\left(\frac{1}{16}\right)^{-1}x^{2 \times (-1)} = 4 \times 16x^{-2}$

$= 64x^{-2}$

Substitute $y = \frac{1}{16}x^2$ into $y^{\frac{1}{2}}$.

$\left(\frac{1}{16}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{16}}$ and $(x^2)^{\frac{1}{2}} = x^{2 \times \frac{1}{2}}$

$\left(\frac{1}{16}\right)^{-1} = 16$ and $x^{2 \times (-1)} = x^{-2}$

Problem-solving

Check that your answers are in the correct form. If k and n are constants they could be positive or negative, and they could be integers, fractions or surds.

Exercise 1D

SKILLS PROBLEM-SOLVING

1 Simplify:

a $x^3 \div x^{-2}$ **b** $x^5 \div x^7$ **c** $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$

d $(x^2)^{\frac{3}{2}}$ **e** $(x^3)^{\frac{5}{3}}$ **f** $3x^{0.5} \times 4x^{-0.5}$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}}$ **h** $5x^{\frac{7}{5}} \div x^{\frac{2}{5}}$ **i** $3x^4 \times 2x^{-5}$

j $\sqrt{x} \times \sqrt[3]{x}$ **k** $(\sqrt{x})^3 \times (\sqrt[3]{x})^4$ **l** $\frac{(\sqrt[3]{x})^2}{\sqrt{x}}$

2 Evaluate, without using your calculator:

a $25^{\frac{1}{2}}$ **b** $81^{\frac{3}{2}}$ **c** $27^{\frac{1}{3}}$

d 4^{-2} **e** $9^{-\frac{1}{2}}$ **f** $(-5)^{-3}$

g $\left(\frac{3}{4}\right)^0$ **h** $1296^{\frac{3}{4}}$ **i** $\left(\frac{25}{16}\right)^{\frac{3}{2}}$

j $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ **k** $\left(\frac{6}{5}\right)^{-1}$ **l** $\left(\frac{343}{512}\right)^{-\frac{2}{3}}$

3 Simplify:

a $(64x^{10})^{\frac{1}{2}}$ **b** $\frac{5x^3 - 2x^2}{x^5}$ **c** $(125x^{12})^{\frac{1}{3}}$ **d** $\frac{x + 4x^3}{x^3}$

e $\frac{2x + x^2}{x^4}$ **f** $\left(\frac{4}{9}x^4\right)^{\frac{3}{2}}$ **g** $\frac{9x^2 - 15x^5}{3x^3}$ **h** $\frac{5x + 3x^2}{15x^3}$

(E) **4** **a** Find the value of $81^{\frac{1}{4}}$. **(1 mark)**

b Simplify $x(2x^{-\frac{1}{3}})^4$. **(2 marks)**

(E) **5** Given that $y = \frac{1}{8}x^3$, express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(2 marks)**

b $\frac{1}{2}y^{-2}$ **(2 marks)**

1.5 Surds

If n is an integer that is *not* a **square number**, then any multiple of $\sqrt{n}$ is called a surd.
Examples of surds are $\sqrt{2}$, $\sqrt{19}$ and $5\sqrt{2}$.

Surds are examples of **irrational** numbers.
The decimal expansion of a surd is never-ending and never repeats, for example $\sqrt{2} = 1.414213562\ldots$

Notation Irrational numbers cannot be written in the form $\frac{a}{b}$ where a and b are integers.

You can use surds to write exact answers to calculations.

■ You can **manipulate** surds using these rules:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example 12

Simplify:

a $\sqrt{12}$ **b** $\frac{\sqrt{20}}{2}$ **c** $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

a $\sqrt{12} = \sqrt{(4 \times 3)}$

$= \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Look for a factor of 12 that is a square number. Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$. $\sqrt{4} = 2$

b $\frac{\sqrt{20}}{2} = \frac{\sqrt{4} \times \sqrt{5}}{2}$

$= \frac{2 \times \sqrt{5}}{2} = \sqrt{5}$

$\sqrt{20} = \sqrt{4} \times \sqrt{5}$

$\sqrt{4} = 2$

Cancel by 2.

c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

$= 5\sqrt{6} - 2\sqrt{6} \times \sqrt{4} + \sqrt{6} \times \sqrt{49}$

$= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49})$

$= \sqrt{6}(5 - 2 \times 2 + 7)$

$= \sqrt{6}(8)$

$= 8\sqrt{6}$

$\sqrt{6}$ is a common factor.

Work out the square roots $\sqrt{4}$ and $\sqrt{49}$.

$5 - 4 + 7 = 8$

Example 13 SKILLS PROBLEM-SOLVING

Expand and simplify if possible:

a $\sqrt{2}(5-\sqrt{3})$ **b** $(2-\sqrt{3})(5+\sqrt{3})$

a $\sqrt{2}(5-\sqrt{3})$ — $\sqrt{2}\times 5-\sqrt{2}\times\sqrt{3}$

$= 5\sqrt{2}-\sqrt{2}\sqrt{3}$

$= 5\sqrt{2}-\sqrt{6}$ — Using $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$

b $(2-\sqrt{3})(5+\sqrt{3})$

$= 2(5+\sqrt{3})-\sqrt{3}(5+\sqrt{3})$ — Expand the brackets completely before you simplify.

$= 10+2\sqrt{3}-5\sqrt{3}-\sqrt{9}$ — Collect like terms: $2\sqrt{3}-5\sqrt{3}=-3\sqrt{3}$

$= 7-3\sqrt{3}$ — Simplify any roots if possible: $\sqrt{9}=3$

Exercise 1E SKILLS PROBLEM-SOLVING

Do not use your calculator for this exercise.

1 Simplify:

a $\sqrt{28}$ **b** $\sqrt{72}$ **c** $\sqrt{50}$

d $\sqrt{32}$ **e** $\sqrt{90}$ **f** $\frac{\sqrt{12}}{2}$

g $\frac{\sqrt{27}}{3}$ **h** $\sqrt{20}+\sqrt{80}$ **i** $\sqrt{200}+\sqrt{18}-\sqrt{72}$

j $\sqrt{175}+\sqrt{63}+2\sqrt{28}$ **k** $\sqrt{28}-2\sqrt{63}+\sqrt{7}$ **l** $\sqrt{80}-2\sqrt{20}+3\sqrt{45}$

m $3\sqrt{80}-2\sqrt{20}+5\sqrt{45}$ **n** $\frac{\sqrt{44}}{\sqrt{11}}$ **o** $\sqrt{12}+3\sqrt{48}+\sqrt{75}$

2 Expand and simplify if possible:

a $\sqrt{3}(2+\sqrt{3})$ **b** $\sqrt{5}(3-\sqrt{3})$ **c** $\sqrt{2}(4-\sqrt{5})$

d $(2-\sqrt{2})(3+\sqrt{5})$ **e** $(2-\sqrt{3})(3-\sqrt{7})$ **f** $(4+\sqrt{5})(2+\sqrt{5})$

g $(5-\sqrt{3})(1-\sqrt{3})$ **h** $(4+\sqrt{3})(2-\sqrt{3})$ **i** $(7-\sqrt{11})(2+\sqrt{11})$

E **3** Simplify $\sqrt{75}-\sqrt{12}$ giving your answer in the form $a\sqrt{3}$, where a is an integer. **(2 marks)**

1.6 Rationalising denominators

If a fraction has a surd in the **denominator**, it is sometimes useful to **rearrange** it so that the denominator is a rational number. This is called rationalising the denominator.

- The rules to rationalise denominators are:
  - For fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
  - For fractions in the form $\frac{1}{a+\sqrt{b}}$, multiply the numerator and denominator by $(a-\sqrt{b})$.
  - For fractions in the form $\frac{1}{a-\sqrt{b}}$, multiply the numerator and denominator by $(a+\sqrt{b})$.

Example 14

Rationalise the denominator:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ **d** $\frac{1}{(1-\sqrt{3})^2}$

a $\frac{1}{\sqrt{3}} = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$ — Multiply the numerator and denominator by $\sqrt{3}$.

$= \frac{\sqrt{3}}{3}$ — $\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

b $\frac{1}{3+\sqrt{2}} = \frac{1 \times (3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$ — Multiply numerator and denominator by $(3-\sqrt{2})$.

$= \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2}$ — $\sqrt{2} \times \sqrt{2} = 2$

$= \frac{3-\sqrt{2}}{7}$ — $9-2=7$, $-3\sqrt{2}+3\sqrt{2}=0$

c $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$ — Multiply numerator and denominator by $(\sqrt{5}+\sqrt{2})$.

$= \frac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$ — $-\sqrt{2}\sqrt{5}$ and $\sqrt{5}\sqrt{2}$ cancel each other out.

$= \frac{7+2\sqrt{10}}{3}$ — $\sqrt{5}\sqrt{2} = \sqrt{10}$

d $\frac{1}{(1-\sqrt{3})^2} = \frac{1}{(1-\sqrt{3})(1-\sqrt{3})}$ — Expand the brackets.

$= \frac{1}{1-\sqrt{3}-\sqrt{3}+\sqrt{9}}$ — Simplify and collect like terms. $\sqrt{9} = 3$

$= \frac{1}{4-2\sqrt{3}}$

$= \frac{1 \times (4+2\sqrt{3})}{(4-2\sqrt{3})(4+2\sqrt{3})}$ — Multiply the numerator and denominator by $(4+2\sqrt{3})$.

— $\sqrt{3} \times \sqrt{3} = 3$

$= \frac{4+2\sqrt{3}}{16+8\sqrt{3}-8\sqrt{3}-12}$ — $16-12=4$, $8\sqrt{3}-8\sqrt{3}=0$

$= \frac{4+2\sqrt{3}}{4} = \frac{2+\sqrt{3}}{2}$

Exercise 1F

SKILLS ANALYSIS

Do not use your calculator for this exercise.

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

E/P **4** Simplify $\frac{3-2\sqrt{5}}{\sqrt{5}-1}$ giving your answer in the form $p+q\sqrt{5}$, where p and q are rational numbers. **(4 marks)**

Problem-solving

You can check that your answer is in the correct form by writing down the values of p and q and checking that they are rational numbers.

Chapter review 1

SKILLS EXECUTIVE FUNCTION

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$ **c** $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand and simplify if possible:

a $(x+3)(x-5)$ **b** $(2x-7)(3x+1)$ **c** $(2x+5)(3x-y+2)$

3 Expand and simplify if possible:

a $x(x+4)(x-1)$ **b** $(x+2)(x-3)(x+7)$ **c** $(2x+3)(x-2)(3x-1)$

4 Expand the brackets:

a $3(5y+4)$ **b** $5x^2(3-5x+2x^2)$ **c** $5x(2x+3)-2x(1-3x)$ **d** $3x^2(1+3x)-2x(3x-2)$

5 Factorise these expressions completely:

a $3x^2 + 4x$ **b** $4y^2 + 10y$ **c** $x^2 + xy + xy^2$ **d** $8xy^2 + 10x^2y$

6 Factorise:

a $x^2 + 3x + 2$ **b** $3x^2 + 6x$ **c** $x^2 - 2x - 35$ **d** $2x^2 - x - 3$

e $5x^2 - 13x - 6$ **f** $6 - 5x - x^2$

7 Factorise:

a $2x^3 + 6x$ **b** $x^3 - 36x$ **c** $2x^3 + 7x^2 - 15x$

8 Simplify:

a $9x^3 \div 3x^{-3}$ **b** $(4^{\frac{3}{2}})^{\frac{1}{3}}$ **c** $3x^{-2} \times 2x^4$ **d** $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

9 Evaluate, without using your calculator:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **b** $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

10 Simplify, without using your calculator:

a $\frac{3}{\sqrt{63}}$ **b** $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

11 **a** Find the value of $35x^2 + 2x - 48$ when $x = 25$.

b By factorising the expression, show that your answer to part **a** can be written as the product of two **prime** factors.

12 Expand and simplify if possible, without using your calculator:

a $\sqrt{2}(3 + \sqrt{5})$ **b** $(2 - \sqrt{5})(5 + \sqrt{3})$ **c** $(6 - \sqrt{2})(4 - \sqrt{7})$

13 Rationalise the denominator and simplify:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2} - 1}$ **c** $\frac{3}{\sqrt{3} - 2}$ **d** $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$ **e** $\frac{1}{(2 + \sqrt{3})^2}$ **f** $\frac{1}{(4 - \sqrt{7})^2}$

14 **Do not use your calculator for this question.**

a Given that $x^3 - x^2 - 17x - 15 = (x + 3)(x^2 + bx + c)$, where b and c are constants, work out the values of b and c.

b **Hence**, fully factorise $x^3 - x^2 - 17x - 15$.

E **15** Given that $y = \frac{1}{64}x^3$, express each of the following in the form kx^n, where k and n are constants.

a $y^{\frac{1}{3}}$ **(1 mark)**

b $4y^{-1}$ **(1 mark)**

E/P **16** Show that $\frac{5}{\sqrt{75} - \sqrt{50}}$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are integers. **(5 marks)**

E **17** Expand and simplify $(\sqrt{11} - 5)(5 - \sqrt{11})$, without using your calculator. **(2 marks)**

E **18** Factorise completely $x - 64x^3$. **(3 marks)**

E/P **19** Express $27^{2x + 1}$ in the form 3^y, stating y in terms of x. **(2 marks)**

E/P **20** Solve the equation $8 + x\sqrt{12} = \frac{8x}{\sqrt{3}}$.

Give your answer in the form $a\sqrt{b}$, where a and b are integers. **(4 marks)**

P **21** **Do not use your calculator for this question.**

A rectangle has a length of $(1 + \sqrt{3})$ cm and area of $\sqrt{12}$ cm^2.

Calculate the width of the rectangle in cm.

Express your answer in the form $a + b\sqrt{3}$, where a and b are integers to be found.

E **22** Show that $\frac{(2 - \sqrt{x})^2}{\sqrt{x}}$ can be written as $4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$. **(2 marks)**

E/P **23** Given that $243\sqrt{3} = 3^a$, find the value of a. **(3 marks)**

E/P **24** Given that $\frac{4x^3 + x^{\frac{5}{2}}}{\sqrt{x}}$ can be written in the form $4x^a + x^b$,

write down the value of a and the value of b. **(2 marks)**

Challenge

a Simplify $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1} + \sqrt{2}} + \frac{1}{\sqrt{2} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{4}} + \ldots + \frac{1}{\sqrt{24} + \sqrt{25}} = 4$

Summary of key points

1 You can use the laws of indices to simplify powers of the same base.

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$

2 Factorising is the opposite of expanding brackets.

3 A quadratic expression has the form $ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

4 $x^2 - y^2 = (x + y)(x - y)$

5 You can use the laws of indices with any rational power.

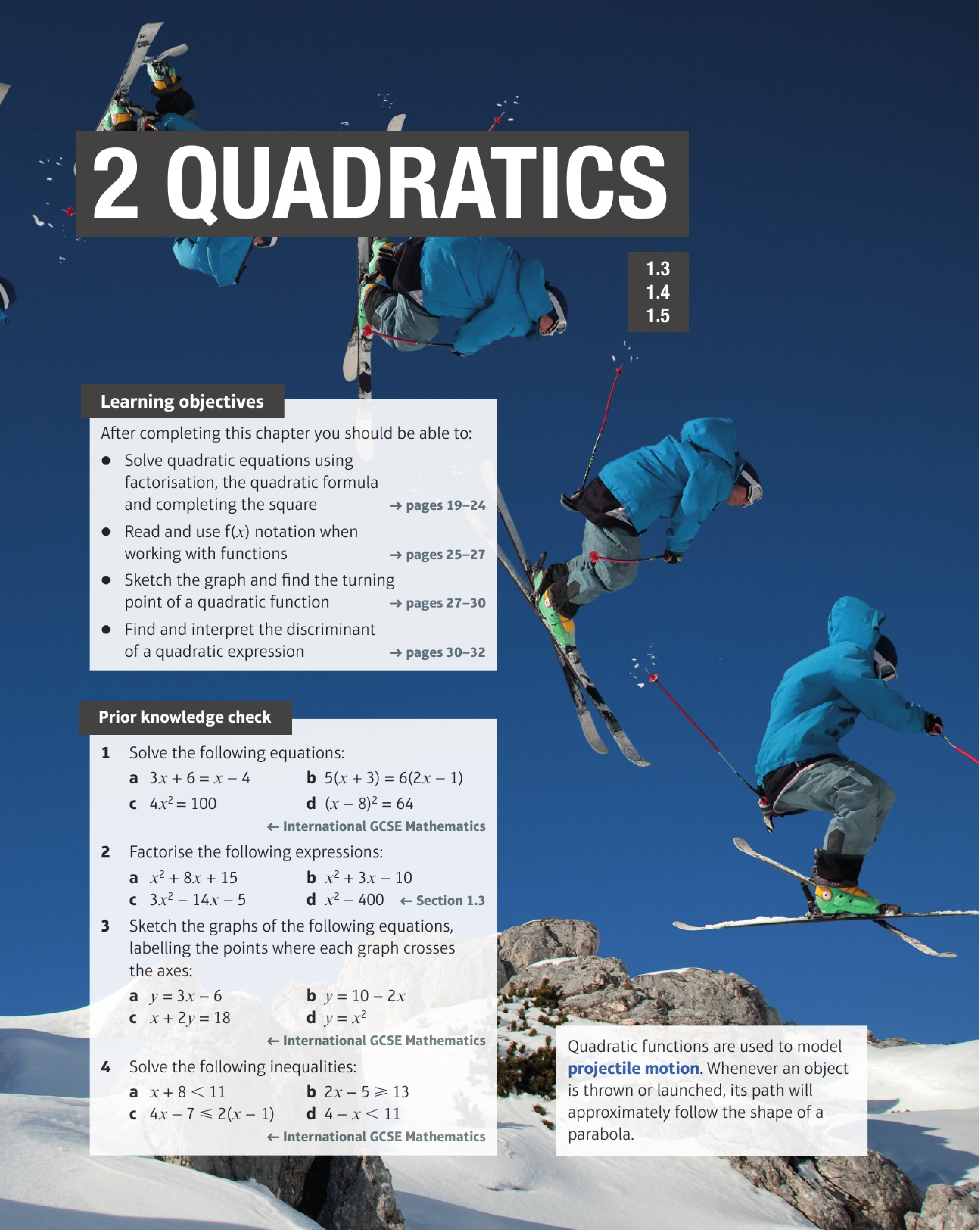
- $a^{\frac{1}{m}} = \sqrt[m]{a}$
- $a^{\frac{n}{m}} = \sqrt[m]{a^n}$
- $a^{-m} = \frac{1}{a^m}$
- $a^0 = 1$

6 You can manipulate surds using these rules:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

7 The rules to rationalise denominators are:

- For fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$.
- For fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $(a - \sqrt{b})$.
- For fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $(a + \sqrt{b})$.



2 QUADRATICS

1.3
1.4
1.5

Learning objectives

After completing this chapter you should be able to:

- Solve quadratic equations using factorisation, the quadratic formula and completing the square → pages 19–24
- Read and use f(x) notation when working with functions → pages 25–27
- Sketch the graph and find the turning point of a quadratic function → pages 27–30
- Find and interpret the discriminant of a quadratic expression → pages 30–32

Prior knowledge check

1 Solve the following equations:

a $3x + 6 = x - 4$ **b** $5(x + 3) = 6(2x - 1)$

c $4x^2 = 100$ **d** $(x - 8)^2 = 64$

← International GCSE Mathematics

2 Factorise the following expressions:

a $x^2 + 8x + 15$ **b** $x^2 + 3x - 10$

c $3x^2 - 14x - 5$ **d** $x^2 - 400$ ← Section 1.3

3 Sketch the graphs of the following equations, labelling the points where each graph crosses the axes:

a $y = 3x - 6$ **b** $y = 10 - 2x$

c $x + 2y = 18$ **d** $y = x^2$

← International GCSE Mathematics

4 Solve the following inequalities:

a $x + 8 < 11$ **b** $2x - 5 \geqslant 13$

c $4x - 7 \leqslant 2(x - 1)$ **d** $4 - x < 11$

← International GCSE Mathematics

Quadratic functions are used to model **projectile motion**. Whenever an object is thrown or launched, its path will approximately follow the shape of a parabola.

2.1 Solving quadratic equations

A quadratic equation can be written in the form $ax^2 + bx + c = 0$, where a, b and c are real constants, and $a \neq 0$. Quadratic equations can have one, two, or no real solutions.

- To solve a quadratic equation by factorising:
  - Write the equation in the form $ax^2 + bx + c = 0$
  - Factorise the left-hand side
  - Set each factor equal to zero and solve to find the value(s) of x

Notation The solutions to an equation are sometimes called the **roots** of the equation.

Example 1 SKILLS CRITICAL THINKING

Solve the following equations:

a $x^2 - 2x - 15 = 0$ **b** $x^2 = 9x$

c $6x^2 + 13x - 5 = 0$ **d** $x^2 - 5x + 18 = 2 + 3x$

a $x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$ — Factorise the quadratic. ← Section 1.3

Then either $x + 3 = 0 \Rightarrow x = -3$ — If the product of the factors is zero, one of the factors must be zero.

or $x - 5 = 0 \Rightarrow x = 5$

Notation The symbol $\Rightarrow$ means 'implies that'. This statement says 'If $x + 3 = 0$, then $x = -3$'.

So $x = -3$ and $x = 5$ are the two solutions of the equation. — A quadratic equation with two **distinct** factors has two distinct solutions.

Watch out The signs of the solutions are **opposite** to the signs of the constant terms in each factor.

b $x^2 = 9x$

$x^2 - 9x = 0$ — Be careful not to divide both sides by x, since x may have the value 0. Instead, rearrange into the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$ — Factorise.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

The solutions are $x = 0$ and $x = 9$.

c $6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$ — Factorise.

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$ — Solutions to quadratic equations do not have to be integers. The quadratic equation $(px + q)(rx + s) = 0$ will have solutions $x = -\frac{q}{p}$ and $x = -\frac{s}{r}$.

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ and $x = -\frac{5}{2}$

d $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$ — Rearrange into the form $ax^2 + bx + c = 0$.

$(x - 4)(x - 4) = 0$ — Factorise.

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Notation When a quadratic equation has exactly one root it is called a **repeated root**. You can also say that the equation has two equal roots.

In some cases it may be more straightforward to solve a quadratic equation without factorising.

Example 2

Solve the following equations:

a $(2x - 3)^2 = 25$ **b** $(x - 3)^2 = 7$

Notation The symbol ± lets you write two statements in one line of working. You say 'plus or minus'.

a $(2x - 3)^2 = 25$

$2x - 3 = \pm 5$ — Take the square root of both sides. Remember $5^2 = (-5)^2 = 25$.

$2x = 3 \pm 5$ — Add 3 to both sides.

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ and $x = -1$

b $(x - 3)^2 = 7$ — Take the square root of both sides.

$x - 3 = \pm\sqrt{7}$

$x = 3 \pm \sqrt{7}$

The solutions are $x = 3 + \sqrt{7}$ and $x = 3 - \sqrt{7}$ — You can leave your answer in surd form.

Exercise 2A SKILLS PROBLEM SOLVING

1 Solve the following equations using factorisation:

a $x^2 + 3x + 2 = 0$ **b** $x^2 + 5x + 4 = 0$ **c** $x^2 + 7x + 10 = 0$ **d** $x^2 - x - 6 = 0$

e $x^2 - 8x + 15 = 0$ **f** $x^2 - 9x + 20 = 0$ **g** $x^2 - 5x - 6 = 0$ **h** $x^2 - 4x - 12 = 0$

2 Solve the following equations using factorisation:

a $x^2 = 4x$ **b** $x^2 = 25x$ **c** $3x^2 = 6x$ **d** $5x^2 = 30x$

e $2x^2 + 7x + 3 = 0$ **f** $6x^2 - 7x - 3 = 0$ **g** $6x^2 - 5x - 6 = 0$ **h** $4x^2 - 16x + 15 = 0$

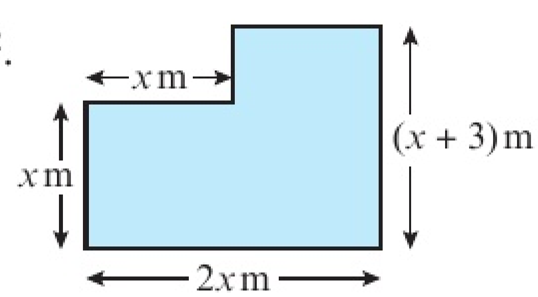
3 Solve the following equations:

a $3x^2 + 5x = 2$ **b** $(2x - 3)^2 = 9$ **c** $(x - 7)^2 = 36$ **d** $2x^2 = 8$ **e** $3x^2 = 5$

f $(x - 3)^2 = 13$ **g** $(3x - 1)^2 = 11$ **h** $5x^2 - 10x^2 = -7 + x + x^2$

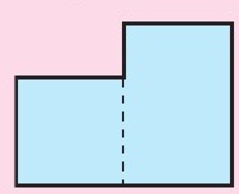
i $6x^2 - 7 = 11x$ **j** $4x^2 + 17x = 6x - 2x^2$

(P) **4** This shape has an area of 44 m². Find the value of x.



Problem-solving

Divide the shape into two sections:



(P) **5** Solve the equation $5x + 3 = \sqrt{3x + 7}$.

Some equations cannot be easily factorised. You can also solve quadratic equations using the **quadratic formula**.

- The solutions of the equation $ax^2 + bx + c = 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Watch out You need to rearrange the equation into the form $ax^2 + bx + c = 0$ before reading off the coefficients.

Notation In $ax^2 + bx + c = 0$, the constants a, b and c are called **coefficients**.

Example 3 SKILLS INTERPRETATION

Solve $3x^2 - 7x - 1 = 0$ by using the quadratic formula.

$3x^2 - 7x - 1 = 0$ — $a = 3$, $b = -7$ and $c = -1$

$x = \frac{-(7) \pm \sqrt{(-7)^2 - 4(3)(-1)}}{2 \times 3}$ — Put brackets around any negative values.

$x = \frac{7 \pm \sqrt{49 + 12}}{6}$ — $-4 \times 3 \times (-1) = +12$

$x = \frac{7 \pm \sqrt{61}}{6}$

Then $x = \frac{7 + \sqrt{61}}{6}$ or $x = \frac{7 - \sqrt{61}}{6}$

Or $x = 2.47$ (3 s.f.) or $x = -0.135$ (3 s.f.)

Exercise 2B SKILLS INTERPRETATION

1 Solve the following equations using the quadratic formula.
Give your answers exactly, leaving them in surd form where necessary.

a $x^2 + 3x + 1 = 0$ **b** $x^2 - 3x - 2 = 0$ **c** $x^2 + 6x + 6 = 0$ **d** $x^2 - 5x - 2 = 0$
e $3x^2 + 10x - 2 = 0$ **f** $4x^2 - 4x - 1 = 0$ **g** $4x^2 - 7x = 2$ **h** $11x^2 + 2x - 7 = 0$

2 Solve the following equations using the quadratic formula.
Give your answers to three **significant figures**.

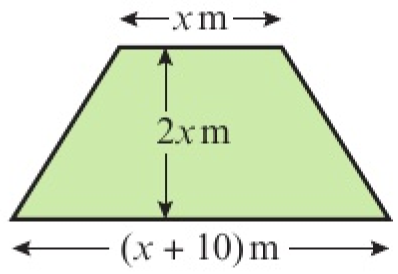
a $x^2 + 4x + 2 = 0$ **b** $x^2 - 8x + 1 = 0$ **c** $x^2 + 11x - 9 = 0$ **d** $x^2 - 7x - 17 = 0$
e $5x^2 + 9x - 1 = 0$ **f** $2x^2 - 3x - 18 = 0$ **g** $3x^2 + 8 = 16x$ **h** $2x^2 + 11x = 5x^2 - 18$

3 For each of the equations below, choose a suitable method and find all of the solutions.
Where necessary, give your answers to three significant figures.

a $x^2 + 8x + 12 = 0$ **b** $x^2 + 9x - 11 = 0$
c $x^2 - 9x - 1 = 0$ **d** $2x^2 + 5x + 2 = 0$
e $(2x + 8)^2 = 100$ **f** $6x^2 + 6 = 12x$
g $2x^2 - 11 = 7x$ **h** $x = \sqrt{8x - 15}$

Hint You can use any method you are confident with to solve these equations.

(P) **4** This trapezium has an area of $50\,\text{m}^2$.
Show that the height of the trapezium is equal to $5(\sqrt{5} - 1)$ m.



Problem-solving

Height must be positive. You will have to **discard** the negative solution of your quadratic equation.

Challenge

Given that x is positive, solve the equation $\frac{1}{x} + \frac{1}{x+2} = \frac{28}{195}$.

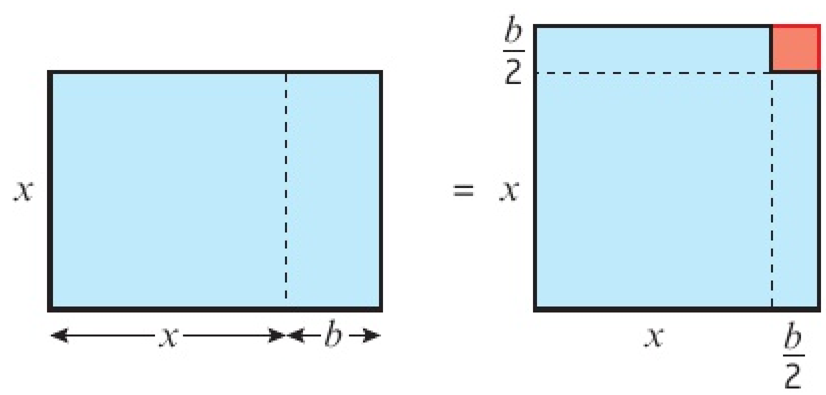
Hint Write the equation in the form $ax^2 + bx + c = 0$ before using the quadratic formula or factorising.

2.2 Completing the square

It is frequently useful to rewrite quadratic expressions by **completing the square**:

- $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

You can draw a diagram of this process when x and b are positive:



$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

The original rectangle has been rearranged into the shape of a square with a smaller square missing. The two areas shaded blue are the same.

Example 4

Complete the square for the expressions:

a $x^2 + 8x$ **b** $x^2 - 3x$ **c** $2x^2 - 12x$

a $x^2 + 8x = (x + 4)^2 - 4^2$
$= (x + 4)^2 - 16$

Begin by halving the coefficient of x.
Using the rule given above, $b = 8$ so $\frac{b}{2} = 4$.

b $x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$
$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$

Be careful if $\frac{b}{2}$ is a fraction. Here $\left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}$.

c $2x^2 - 12x = 2(x^2 - 6x)$
$= 2((x - 3)^2 - 3^2)$
$= 2((x - 3)^2 - 9)$
$= 2(x - 3)^2 - 18$

Here the coefficient of x^2 is 2, so first take out a factor of 2. The other factor is in the form $(x^2 + bx)$ so you can use the rule to complete the square.

Expand the outer bracket by multiplying 2 by 9 to get your answer in this form.

Notation A quadratic expression in the form $p(x + q)^2 + r$ where p, q and r are real constants is in **completed square form**.

- $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

Example 5

Write $3x^2 + 6x + 1$ in the form $p(x + q)^2 + r$, where p, q and r are integers to be found.

$3x^2 + 6x + 1$
$= 3(x^2 + 2x) + 1$
$= 3((x + 1)^2 - 1^2) + 1$
$= 3(x + 1)^2 - 3 + 1$
$= 3(x + 1)^2 - 2$
So $p = 3$, $q = 1$ and $r = -2$.

Watch out This is an **expression**, so you can't divide every term by 3 without changing its value. Instead, you need to take a factor of 3 out of $3x^2 + 6x$.

You could also use the rule given above to complete the square for this expression, but it is safer to learn the method shown here.

Exercise 2C

SKILLS INTERPRETATION

Hint In question **3d**, write the expression as $-4x^2 - 16x + 10$ then take a factor of -4 out of the first two terms to get $-4(x^2 + 4x) + 10$.

1 Complete the square for these expressions:
a $x^2 + 4x$ **b** $x^2 - 6x$ **c** $x^2 - 16x$ **d** $x^2 + x$ **e** $x^2 - 14x$

2 Complete the square for these expressions:
a $2x^2 + 16x$ **b** $3x^2 - 24x$ **c** $5x^2 + 20x$ **d** $2x^2 - 5x$ **e** $8x - 2x^2$

3 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:
a $2x^2 + 8x + 1$ **b** $5x^2 - 15x + 3$ **c** $3x^2 + 2x - 1$ **d** $10 - 16x - 4x^2$ **e** $2x - 8x^2 + 10$

(E) **4** Given that $x^2 + 3x + 6 = (x + a)^2 + b$, find the values of the constants a and b. **(2 marks)**

(E) **5** Write $2 + 0.8x - 0.04x^2$ in the form $A - B(x + C)^2$, where A, B and C are constants **to be determined**. **(3 marks)**

Example 6

SKILLS ANALYSIS

Solve the equation $x^2 + 8x + 10 = 0$ by completing the square.
Give your answers in surd form.

$x^2 + 8x + 10 = 0$ — Check coefficient of $x^2 = 1$.
$x^2 + 8x = -10$ — Subtract 10 to get the LHS in the form $x^2 + bx$.
$(x + 4)^2 - 4^2 = -10$ — Complete the square for $x^2 + 8x$.
$(x + 4)^2 = -10 + 16$ — Add 4^2 to both sides.
$(x + 4)^2 = 6$
$x + 4 = \pm\sqrt{6}$ — Take the square root of both sides.
$x = -4 \pm \sqrt{6}$ — Subtract 4 from both sides.
So the solutions are
$x = -4 + \sqrt{6}$ and $x = -4 - \sqrt{6}$. — Leave your answer in surd form.

Example 7

Solve the equation $2x^2 - 8x + 7 = 0$. Give your answers in surd form.

$$2x^2 - 8x + 7 = 0$$
$$x^2 - 4x + \frac{7}{2} = 0$$
$$x^2 - 4x = -\frac{7}{2}$$
$$(x - 2)^2 - 2^2 = -\frac{7}{2}$$
$$(x - 2)^2 = -\frac{7}{2} + 4$$
$$(x - 2)^2 = \frac{1}{2}$$
$$x - 2 = \pm\sqrt{\frac{1}{2}}$$
$$x = 2 \pm \frac{1}{\sqrt{2}}$$

So the roots are

$x = 2 + \frac{1}{\sqrt{2}}$ and $x = 2 - \frac{1}{\sqrt{2}}$.

Problem-solving

This is an **equation** so you can divide every term by the same constant. Divide by 2 to get x^2 on its own. The right-hand side is 0 so it is unchanged.

Complete the square for $x^2 - 4x$.

Add 2^2 to both sides.

Take the square root of both sides.

Add 2 to both sides.

Online Use your calculator to check solutions to quadratic equations quickly.

Exercise 2D SKILLS ANALYSIS

1 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $x^2 + 6x + 1 = 0$ **b** $x^2 + 12x + 3 = 0$ **c** $x^2 + 4x - 2 = 0$ **d** $x^2 - 10x = 5$

2 Solve these quadratic equations by completing the square. Leave your answers in surd form.

a $2x^2 + 6x - 3 = 0$ **b** $5x^2 + 8x - 2 = 0$ **c** $4x^2 - x - 8 = 0$ **d** $15 - 6x - 2x^2 = 0$

(E) 3 $x^2 - 14x + 1 = (x + p)^2 + q$, where p and q are constants.

a Find the values of p and q. **(2 marks)**

b Using your answer to part **a**, or otherwise, show that the solutions to the equation $x^2 - 14x + 1 = 0$ can be written in the form $r \pm s\sqrt{3}$, where r and s are constants to be found. **(2 marks)**

(E/P) 4 By completing the square, show that the solutions to the equation $x^2 + 2bx + c = 0$ are given by the formula $x = -b \pm \sqrt{b^2 - c}$. **(4 marks)**

Problem-solving

Follow the same steps as you would if the coefficients were numbers.

Challenge

a Show that the solutions to the equation

$ax^2 + 2bx + c = 0$ are given by $x = -\frac{b}{a} \pm \sqrt{\frac{b^2 - ac}{a^2}}$.

b Hence, or otherwise, show that the solutions to the equation $ax^2 + bx + c = 0$ can be written as

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Hint Start by dividing the whole equation by a.

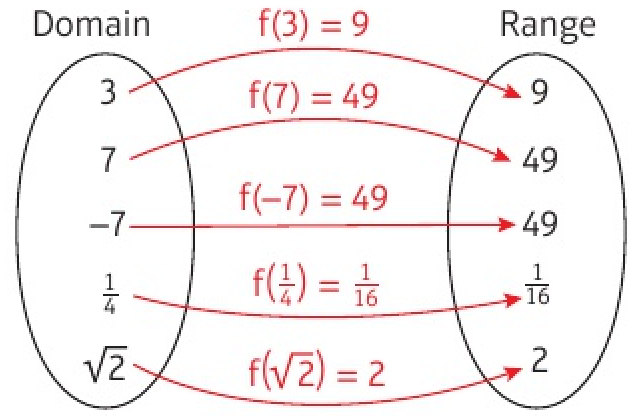
Links You can use this method to prove the quadratic formula. ← Section 2.1

2.3 Functions

A function is a mathematical relationship that maps each value of a **set** of inputs to a single output. The notation $f(x)$ is used to represent a function of x.

- The set of possible inputs for a function is called the **domain**.
- The set of possible outputs of a function is called the **range**.

This diagram shows how the function $f(x) = x^2$ maps five values in its domain to values in its range.



- The roots of a function are the values of x for which $f(x) = 0$.

Example 8

The functions f and g are given by $f(x) = 2x - 10$ and $g(x) = x^2 - 9$, $x \in \mathbb{R}$.

a Find the values of $f(5)$ and $g(10)$.

b Find the value of x for which $f(x) = g(x)$.

Notation If the input of a function, x, can be any real number, then the domain can be written as $x \in \mathbb{R}$.

The symbol $\in$ means 'is a **member** of' and the symbol $\mathbb{R}$ represents the set of real numbers.

a $f(5) = 2(5) - 10 = 10 - 10 = 0$

$g(10) = (10)^2 - 9 = 100 - 9 = 91$

To find f(5), substitute $x = 5$ into the function $f(x)$.

b $f(x) = g(x)$

$2x - 10 = x^2 - 9$

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

$x = 1$

Set $f(x)$ equal to $g(x)$ and solve for x.

$x = 1$ is a repeated root.

Example 9

The function f is defined as $f(x) = x^2 + 6x - 5$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(x + p)^2 + q$.

b Hence, or otherwise, find the roots of $f(x)$, leaving your answers in surd form.

c Write down the minimum value of $f(x)$, and state the value of x for which it occurs.

a $f(x) = x^2 + 6x - 5$

$= (x + 3)^2 - 9 - 5$

$= (x + 3)^2 - 14$

Complete the square for $x^2 + 6x$ and then simplify the expression.

b $f(x) = 0$

To find the root(s) of a function, set it equal to zero.

$(x + 3)^2 - 14 = 0$

$(x + 3)^2 = 14$

You can solve this equation directly. Remember to write $\pm$ when you take square roots of both sides.

$x + 3 = \pm\sqrt{14}$

$x = -3 \pm \sqrt{14}$

$f(x)$ has two roots:

$-3 + \sqrt{14}$ and $-3 - \sqrt{14}$.

c $(x+3)^2 \geqslant 0$
So the minimum value of $f(x)$ is -14.
This occurs when $(x+3)^2 = 0$,
so when $x = -3$.

A squared value must be greater than or equal to 0.

$(x+3)^2 \geqslant 0$ so $(x+3)^2 - 14 \geqslant -14$

Example 10

Find the roots of the function $f(x) = x^6 + 7x^3 - 8$, $x \in \mathbb{R}$.

$f(x) = 0$
$x^6 + 7x^3 - 8 = 0$
$(x^3)^2 + 7(x^3) - 8 = 0$
$(x^3 - 1)(x^3 + 8) = 0$
So $x^3 = 1$ or $x^3 = -8$
$x^3 = 1 \Rightarrow x = 1$
$x^3 = -8 \Rightarrow x = -2$
The roots of $f(x)$ are 1 and -2.

Alternatively, let $u = x^3$.
$f(x) = x^6 + 7x^3 - 8$
$= (x^3)^2 + 7(x^3) - 8$
$= u^2 + 7u - 8$
$= (u - 1)(u + 8)$
So when $f(x) = 0$, $u = 1$ or $u = -8$.
If $u = 1 \Rightarrow x^3 = 1 \Rightarrow x = 1$
If $u = -8 \Rightarrow x^3 = -8 \Rightarrow x = -2$
The roots of $f(x)$ are 1 and -2.

Problem-solving
$f(x)$ can be written as a function of a function. The only powers of x in $f(x)$ are 6, 3 and 0 so you can write it as a quadratic function of x^3.

Treat x^3 as a single variable and factorise.

Solve the quadratic equation to find two values for x^3, then find the corresponding values of x.

You can simplify this working with a substitution.

Replace x^3 with u then solve the quadratic equation in u.

Watch out The solutions to the quadratic equation will be values of u. Convert back to values of x using your substitution.

Exercise 2E

SKILLS INTERPRETATION

1 Using the functions $f(x) = 5x + 3$, $g(x) = x^2 - 2$ and $h(x) = \sqrt{x+1}$, find the values of:

a $f(1)$ **b** $g(3)$ **c** $h(8)$ **d** $f(1.5)$ **e** $g(\sqrt{2})$

f $h(-1)$ **g** $f(4) + g(2)$ **h** $f(0) + g(0) + h(0)$ **i** $\dfrac{g(4)}{h(3)}$

(P) **2** The function $f(x)$ is defined by $f(x) = x^2 - 2x$, $x \in \mathbb{R}$. Given that $f(a) = 8$, find two possible values for a.

Problem-solving
Substitute $x = a$ into the function and set the resulting expression equal to 8.

3 Find all the roots of the following functions:

a $f(x) = 10 - 15x$ **b** $g(x) = (x + 9)(x - 2)$ **c** $h(x) = x^2 + 6x - 40$

d $j(x) = 144 - x^2$ **e** $k(x) = x(x + 5)(x + 7)$ **f** $m(x) = x^3 + 5x^2 - 24x$

4 The functions p and q are given by $p(x) = x^2 - 3x$ and $q(x) = 2x - 6$, $x \in \mathbb{R}$.
Find the two values of x for which $p(x) = q(x)$.

5 The functions f and g are given by $f(x) = 2x^3 + 30x$ and $g(x) = 17x^2$, $x \in \mathbb{R}$.
Find the three values of x for which $f(x) = g(x)$.

(E) **6** The function f is defined as $f(x) = x^2 - 2x + 2$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(x + p)^2 + q$, where p and q are constants to be found. **(2 marks)**

b Hence, or otherwise, explain why $f(x) > 0$ for all values of x, and find the minimum value of $f(x)$. **(1 mark)**

7 Find all roots of the following functions:

a $f(x) = x^6 + 9x^3 + 8$
b $g(x) = x^4 - 12x^2 + 32$
c $h(x) = 27x^6 + 26x^3 - 1$
d $j(x) = 32x^{10} - 33x^5 + 1$
e $k(x) = x - 7\sqrt{x} + 10$
f $m(x) = 2x^{\frac{2}{3}} + 2x^{\frac{1}{3}} - 12$

Hint The function in part **b** has four roots.

(E/P) **8** The function f is defined as $f(x) = 3^{2x} - 28(3^x) + 27$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(3^x - a)(3^x - b)$, where a and b are real constants. **(2 marks)**

b Hence find the two roots of $f(x)$. **(2 marks)**

Problem-solving **Consider** $f(x)$ as a function of a function.

2.4 Quadratic graphs

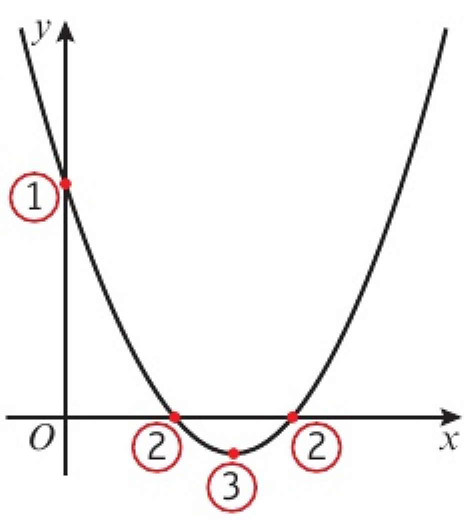
When $f(x) = ax^2 + bx + c$, the **graph** of $y = f(x)$ has a curved shape called a parabola.

You can **sketch** a quadratic graph by identifying key features.

The coefficient of x^2 determines the overall shape of the graph.

When a is positive, the parabola will have this shape: ∪

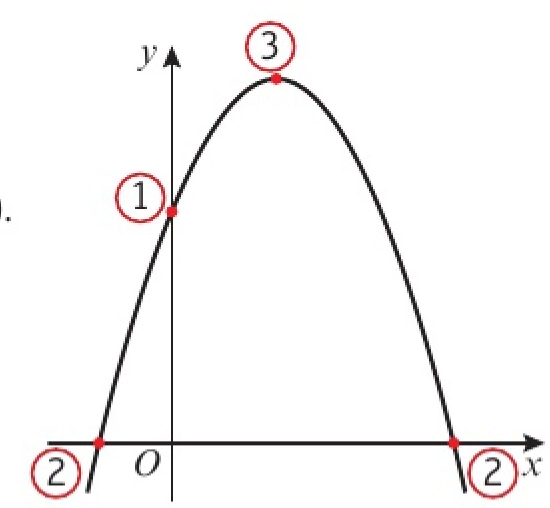
When a is negative, the parabola will have this shape: ∩



(1) The graph crosses the y-**axis** when $x = 0$.
The y-coordinate is equal to c.

(2) The graph crosses the x-axis when $y = 0$.
The x-coordinates are roots of the function $f(x)$.

(3) Quadratic graphs have one turning point.
This can be a minimum or a maximum.
Since a parabola is **symmetrical**, the turning point and line of **symmetry** are half-way between the two roots.



- You can find the coordinates of the turning point of a quadratic graph by completing the square.
If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

Links The graph of $y = a(x + p)^2 + q$ is a translation of the graph of $y = ax^2$ by $\begin{pmatrix} -p \\ q \end{pmatrix}$. → Section 4.4

Example 11 SKILLS INTERPRETATION

Sketch the graph of $y = x^2 - 5x + 4$, and find the coordinates of its turning point.

As $a = 1$ is positive, the graph has a ⋃ shape and a minimum point.

When $x = 0$, $y = 4$, so the graph crosses the y-axis at (0, 4).

When $y = 0$,

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$x = 1$ or $x = 4$, so the graph crosses the x-axis at (1, 0) and (4, 0).

Completing the square:

$$x^2 - 5x + 4 = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 4$$

$$= \left(x - \frac{5}{2}\right)^2 - \frac{9}{4}$$

So the minimum point has coordinates $\left(\frac{5}{2}, -\frac{9}{4}\right)$.

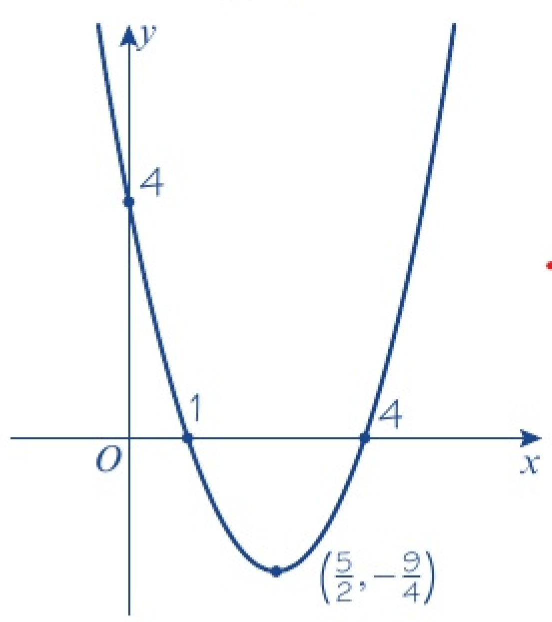
Alternatively, the minimum occurs when x is half-way between 1 and 4,

so $x = \frac{1 + 4}{2} = \frac{5}{2}$

$$y = \left(\frac{5}{2}\right)^2 - 5 \times \left(\frac{5}{2}\right) + 4 = -\frac{9}{4}$$

so the minimum has coordinates $\left(\frac{5}{2}, -\frac{9}{4}\right)$.

The sketch of the graph is:



Use the coefficient of x^2 to **determine** the general shape of the graph.

This example factorises, but you may need to use the quadratic formula or complete the square.

Complete the square to find the coordinates of the turning point.

Watch out If you use symmetry to find the x-coordinate of the minimum point, you need to substitute this value into the equation to find the y-coordinate of the minimum point.

You could use a graphic calculator or substitute some values to check your sketch.

When $x = 5$, $y = 5^2 - 5 \times 5 + 4 = 4$.

Online **Explore** how the graph of $y = (x + p)^2 + q$ changes as the values of p and q change using technology.

Example 12

Sketch the graph of $y = 4x - 2x^2 - 3$. Find the coordinates of its turning point and write down the equation of its line of symmetry.

As $a = -2$ is negative, the graph has a ∩ shape and a maximum point.

It's easier to see that $a < 0$ if you write the equation in the form $y = -2x^2 + 4x - 3$.

When $x = 0$, $y = -3$, so the graph crosses the y-axis at (0, −3).

When $y = 0$,

$-2x^2 + 4x - 3 = 0$.

Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-3)}}{2 \times (-2)}$$

$a = -2$, $b = 4$ and $c = -3$

$$x = \frac{-4 \pm \sqrt{-8}}{-4}$$

You would need to square root a negative number to evaluate this expression. Therefore this equation has no real solutions.

There are no real solutions, so the graph does not cross the x-axis.

Completing the square:

$-2x^2 + 4x - 3$

$= -2(x^2 - 2x) - 3$

The coefficient of x^2 is −2, so take out a factor of −2.

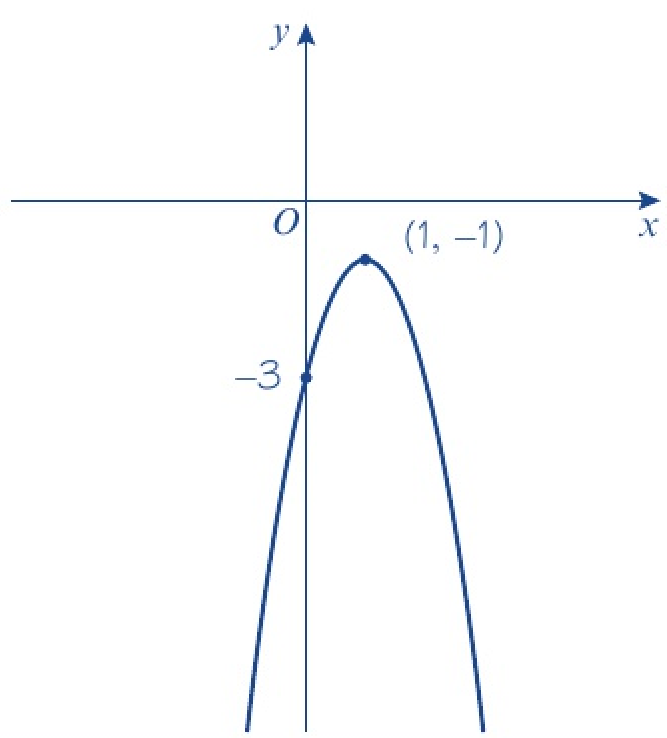
$= -2((x - 1)^2 - 1) - 3$

$= -2(x - 1)^2 + 2 - 3$

$= -2(x - 1)^2 - 1$

So the maximum point has coordinates (1, −1).

The line of symmetry is vertical and goes through the maximum point. It has the equation $x = 1$.



Watch out A sketch graph does not need to be plotted exactly or drawn to scale. However you should:

- draw a smooth curve by hand
- identify any relevant key points (such as intercepts and turning points)
- **label** your **axes**.

Exercise 2F SKILLS ANALYSIS

1 Sketch the graphs of the following equations. For each graph, show the coordinates of the point(s) where the graph crosses the coordinate axes, and write down the coordinates of the turning point and the equation of the line of symmetry.

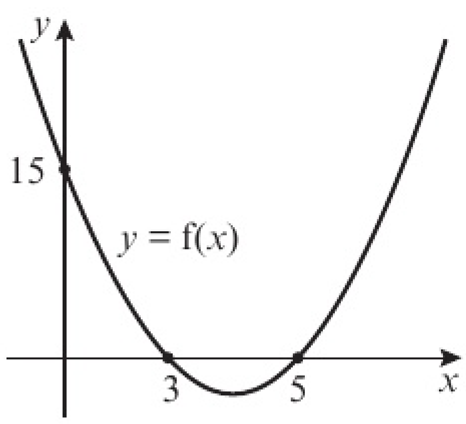
a $y = x^2 - 6x + 8$ **b** $y = x^2 + 2x - 15$ **c** $y = 25 - x^2$ **d** $y = x^2 + 3x + 2$

e $y = -x^2 + 6x + 7$ **f** $y = 2x^2 + 4x + 10$ **g** $y = 2x^2 + 7x - 15$ **h** $y = 6x^2 - 19x + 10$

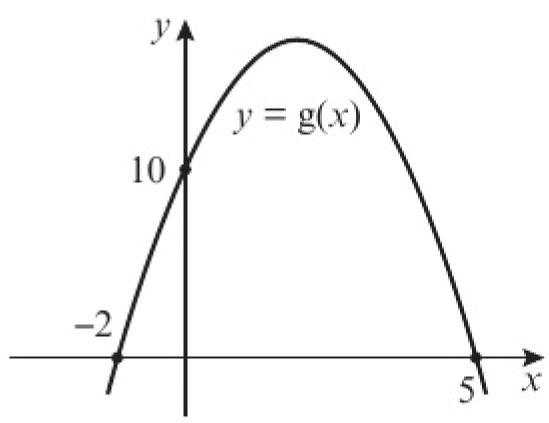
i $y = 4 - 7x - 2x^2$ **j** $y = 0.5x^2 + 0.2x + 0.02$

(P) 2 These sketches are graphs of quadratic functions of the form $ax^2 + bx + c$. Find the values of a, b and c for each function.

a



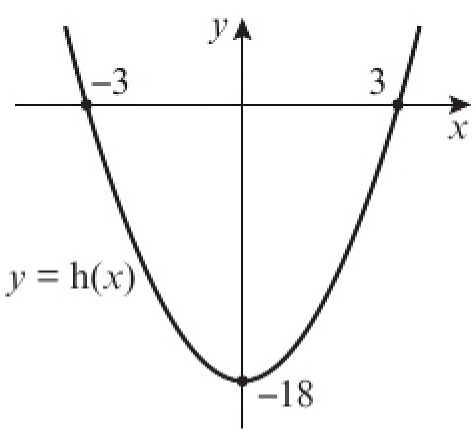
b



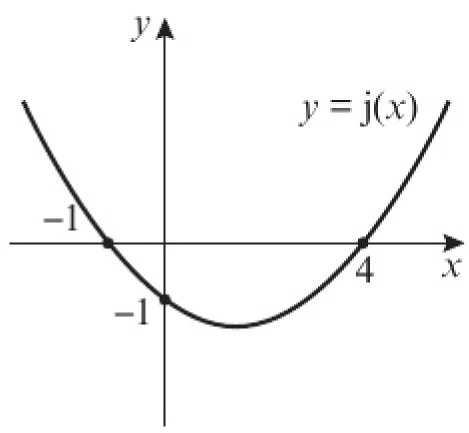
Problem-solving

Check your answers by substituting values into the function. In part **c** the graph passes through (0, −18), so h(0) should be −18.

c



d



(E/P) 3 The graph of $y = ax^2 + bx + c$ has a minimum at $(5, -3)$ and passes through $(4, 0)$. Find the values of a, b and c. **(3 marks)**

2.5 The discriminant

If you square any real number, the result is greater than or equal to 0.
This means that if y is negative, $\sqrt{y}$ cannot be a real number. Look at the quadratic formula:

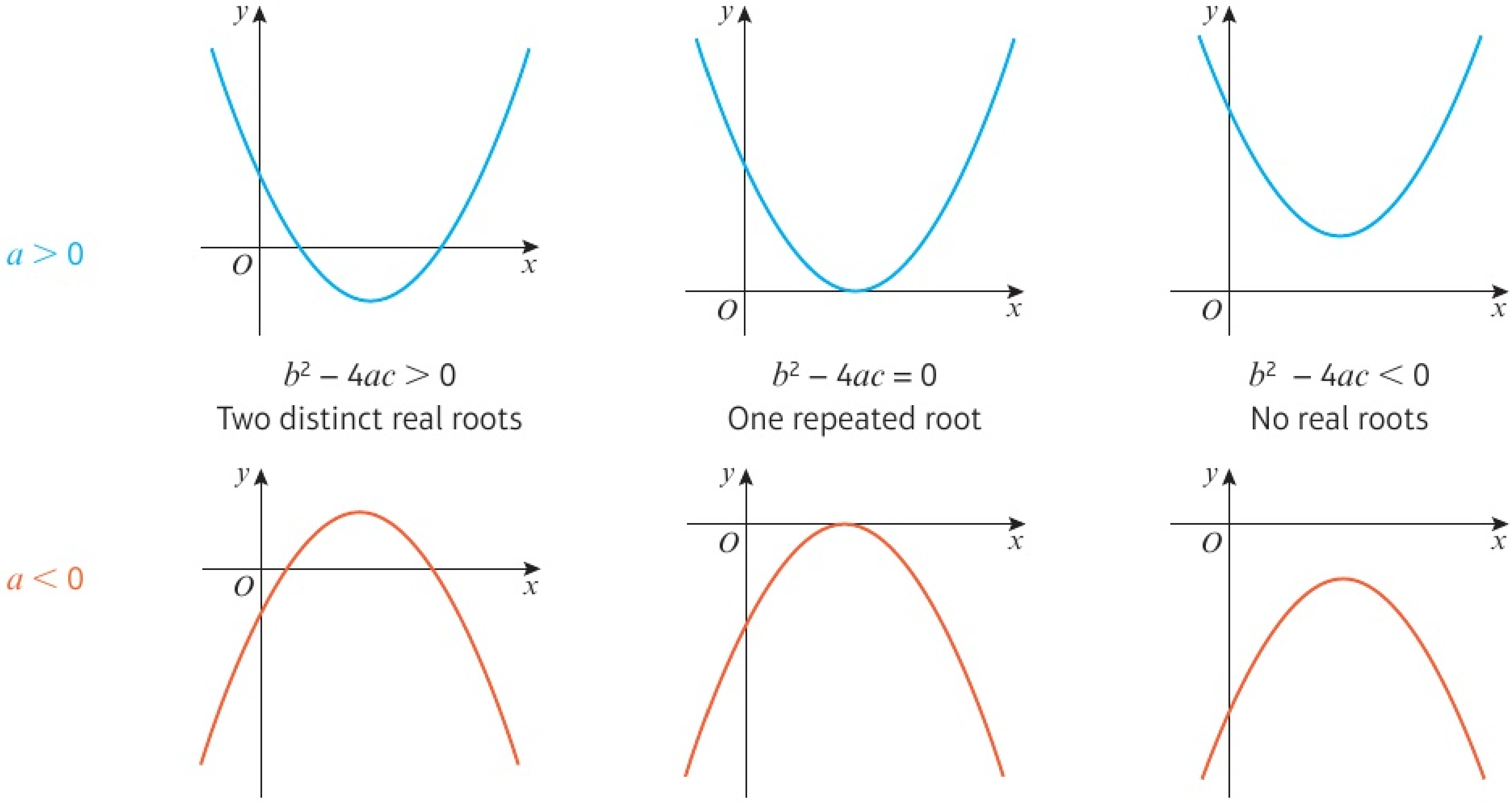
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the value under the square root sign is negative, x cannot be a real number and there are no real solutions. If the value under the square root is equal to 0, both solutions will be the same.

- For the quadratic function $f(x) = ax^2 + bx + c$, the expression $b^2 - 4ac$ is called the discriminant. The value of the discriminant shows how many roots $f(x)$ has:
  - If $b^2 - 4ac > 0$ then $f(x)$ has two distinct real roots.
  - If $b^2 - 4ac = 0$ then $f(x)$ has one repeated root.
  - If $b^2 - 4ac < 0$ then $f(x)$ has no real roots.

You can use the discriminant to check the shape of sketch graphs.

Below are some graphs of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = ax^2 + bx + c$.



Example 13 SKILLS PROBLEM SOLVING

Find the values of k for which $\mathrm{f}(x) = x^2 + kx + 9$ has equal roots.

$x^2 + kx + 9 = 0$
Here $a = 1$, $b = k$ and $c = 9$
For equal roots, $b^2 - 4ac = 0$.
$k^2 - 4 \times 1 \times 9 = 0$
$k^2 - 36 = 0$
$k^2 = 36$
so $k = \pm 6$

Problem-solving
Use the condition given in the question to write a statement about the discriminant.

Substitute for a, b and c to get an equation with one unknown.

Solve to find the values of k.

Example 14

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

$x^2 + 4x + k = 0$
Here $a = 1$, $b = 4$ and $c = k$.
For two real solutions, $b^2 - 4ac > 0$.
$4^2 - 4 \times 1 \times k > 0$
$16 - 4k > 0$
$16 > 4k$
$4 > k$
So $k < 4$

This statement involves an **inequality**, so your answer will also be an inequality.

For any value of k less than 4, the equation will have two distinct real solutions.

Online Explore how the value of the discriminant changes with k using technology.

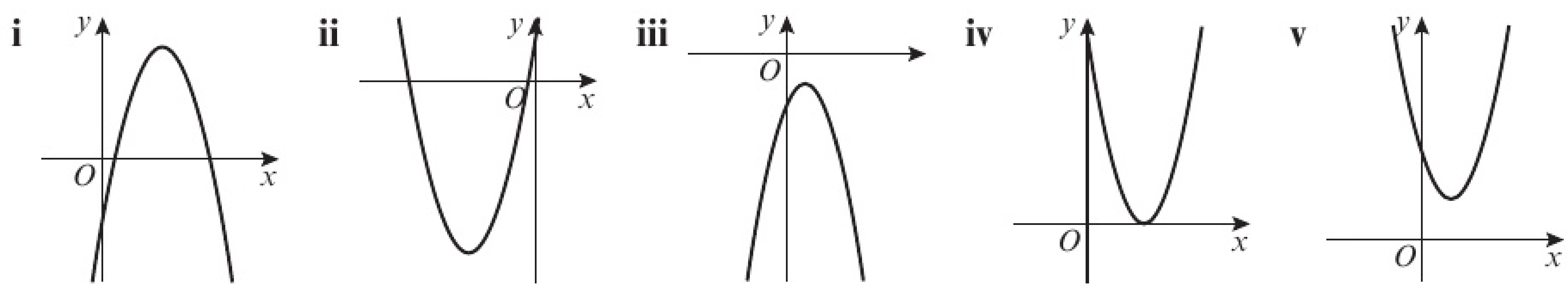
Exercise 2G SKILLS INTERPRETATION

1 a Calculate the value of the discriminant for each of these five functions:

i $f(x) = x^2 + 8x + 3$ ii $g(x) = 2x^2 - 3x + 4$ iii $h(x) = -x^2 + 7x - 3$

iv $j(x) = x^2 - 8x + 16$ v $k(x) = 2x - 3x^2 - 4$

b Using your answers to part **a**, match the same five functions to these sketch graphs.



E/P 2 Find the values of k for which $x^2 + 6x + k = 0$ has two real solutions. **(2 marks)**

E/P 3 Find the value of t for which $2x^2 - 3x + t = 0$ has exactly one solution. **(2 marks)**

E/P 4 Given that the function $f(x) = sx^2 + 8x + s$ has equal roots, find the value of the positive constant s. **(2 marks)**

E/P 5 Find the range of values of k for which $3x^2 - 4x + k = 0$ has no real solutions. **(2 marks)**

E/P 6 The function $g(x) = x^2 + 3px + (14p - 3)$, where p is an integer, has two equal roots.

a Find the value of p. **(2 marks)**

b For this value of p, solve the equation $x^2 + 3px + (14p - 3) = 0$. **(2 marks)**

E/P 7 $h(x) = 2x^2 + (k + 4)x + k$, where k is a real constant.

a Find the discriminant of $h(x)$ in terms of k. **(3 marks)**

b Hence or otherwise, prove that $h(x)$ has two distinct real roots for all values of k. **(3 marks)**

Problem-solving

If a question part says 'hence or otherwise' it is usually easier to use your answer to the previous question part.

Challenge

a Prove that, if the values of a and c are given and non-zero, it is always possible to choose a value of b so that $f(x) = ax^2 + bx + c$ has distinct real roots.

b Is it always possible to choose a value of b so that $f(x)$ has equal roots? Explain your answer.

Chapter review 2

SKILLS EXECUTIVE FUNCTION

1 Solve the following equations without a calculator. Leave your answers in surd form where necessary.

a $y^2 + 3y + 2 = 0$ **b** $3x^2 + 13x - 10 = 0$

c $5x^2 - 10x = 4x + 3$ **d** $(2x - 5)^2 = 7$

2 Sketch graphs of the following equations:

a $y = x^2 + 5x + 4$ **b** $y = 2x^2 + x - 3$

c $y = 6 - 10x - 4x^2$ **d** $y = 15x - 2x^2$

E **3** $f(x) = x^2 + 3x - 5$ and $g(x) = 4x + k$, where k is a constant.

a Given that $f(3) = g(3)$, find the value of k. **(3 marks)**

b Find the values of x for which $f(x) = g(x)$. **(3 marks)**

4 Solve the following equations, giving your answers correct to 3 significant figures:

a $k^2 + 11k - 1 = 0$ **b** $2t^2 - 5t + 1 = 0$

c $10 - x - x^2 = 7$ **d** $(3x - 1)^2 = 3 - x^2$

5 Write each of these expressions in the form $p(x + q)^2 + r$, where p, q and r are constants to be found:

a $x^2 + 12x - 9$ **b** $5x^2 - 40x + 13$

c $8x - 2x^2$ **d** $3x^2 - (x + 1)^2$

E **6** Find the value k for which the equation $5x^2 - 2x + k = 0$ has exactly one solution. **(2 marks)**

E **7** Given that for all values of x:

$$3x^2 + 12x + 5 = p(x + q)^2 + r$$

a find the values of p, q and r. **(3 marks)**

b Hence solve the equation $3x^2 + 12x + 5 = 0$. **(2 marks)**

E/P **8** The function f is defined as $f(x) = 2^{2x} - 20(2^x) + 64$, $x \in \mathbb{R}$.

a Write $f(x)$ in the form $(2^x - a)(2^x - b)$, where a and b are real constants. **(2 marks)**

b Hence find the two roots of $f(x)$. **(2 marks)**

9 Find, as surds, the roots of the equation $2(x + 1)(x - 4) - (x - 2)^2 = 0$.

10 Use algebra to solve $(x - 1)(x + 2) = 18$.

E/P **11** A diver launches herself off a springboard. The height of the diver, in metres, above the pool t seconds after launch can be modelled by the following function:

$$h(t) = 5t - 10t^2 + 10,\ t \geqslant 0$$

a How high is the springboard above the water? **(1 mark)**

b Use the model to find the time at which the diver hits the water. **(3 marks)**

c Rearrange $h(t)$ into the form $A - B(t - C)^2$ and give the values of the constants A, B and C. **(3 marks)**

d Using your answer to part **c**, or otherwise, find the maximum height of the diver, and the time at which this maximum height is reached. **(2 marks)**

E/P **12** For this question, $f(x) = 4kx^2 + (4k + 2)x + 1$, where k is a real constant.

a Find the discriminant of $f(x)$ in terms of k. **(3 marks)**

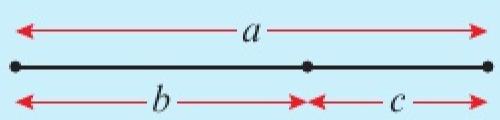
b By simplifying your answer to part **a**, or otherwise, prove that $f(x)$ has two distinct real roots for all non-zero values of k. **(2 marks)**

c Explain why $f(x)$ cannot have two distinct real roots when $k = 0$. **(1 mark)**

E/P **13** Find all of the roots of the function $r(x) = x^8 - 17x^4 + 16$. **(5 marks)**

Challenge

a The ratio of the lengths $a : b$ in this line is the same as the ratio of the lengths $b : c$.



Show that this ratio is $\frac{1 + \sqrt{5}}{2} : 1$.

b Show also that the infinite square root

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}} = \frac{1 + \sqrt{5}}{2}.$$

Summary of key points

1 To solve a quadratic equation by factorising:

- Write the equation in the form $ax^2 + bx + c = 0$
- Factorise the left-hand side
- Set each factor equal to zero and solve to find the value(s) of x

2 The solutions of the equation $ax^2 + bx + c = 0$ where $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

4 $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

5 The set of possible inputs of a function is called the **domain**.

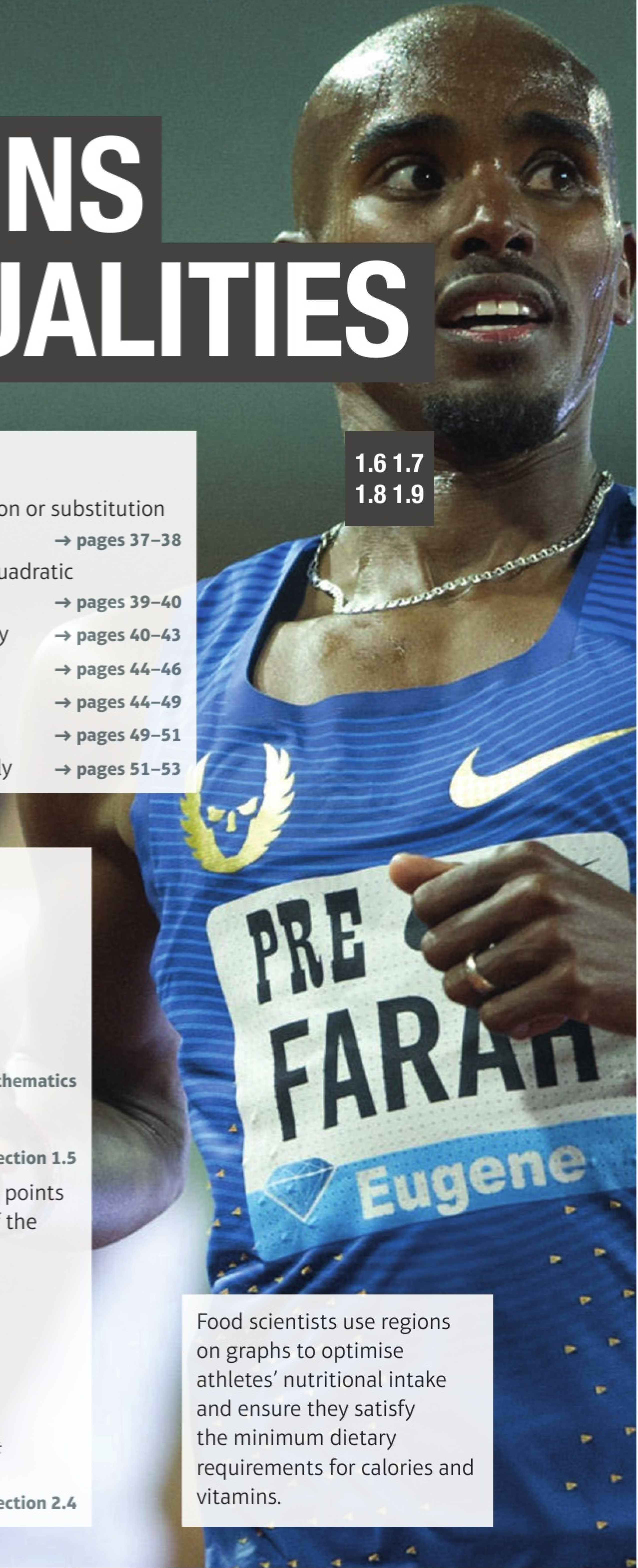
The set of possible outputs of a function is called the **range**.

6 The **roots** of a function are the values of x for which $f(x) = 0$.

7 You can find the coordinates of a **turning point** of a quadratic graph by completing the square. If $f(x) = a(x + p)^2 + q$, the graph of $y = f(x)$ has a turning point at $(-p, q)$.

8 For the quadratic function $f(x) = ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**. The value of the discriminant shows how many roots $f(x)$ has:

- If $b^2 - 4ac > 0$ then the quadratic function has two distinct real roots.
- If $b^2 - 4ac = 0$ then the quadratic function has one repeated real root.
- If $b^2 - 4ac < 0$ then the quadratic function has no real roots.



3 EQUATIONS AND INEQUALITIES

1.6 1.7 1.8 1.9

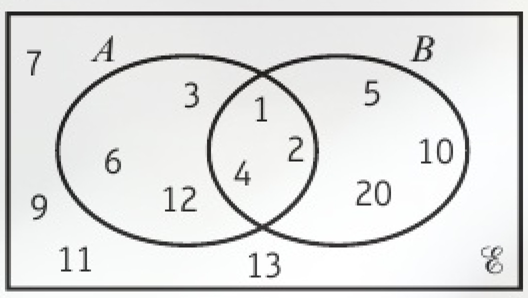
Learning objectives

After completing this chapter you should be able to:

- Solve linear simultaneous equations using elimination or substitution → pages 37–38
- Solve simultaneous equations: one linear and one quadratic → pages 39–40
- Interpret algebraic solutions of equations graphically → pages 40–43
- Solve linear inequalities → pages 44–46
- Solve quadratic inequalities → pages 44–49
- Interpret inequalities graphically → pages 49–51
- Represent linear and quadratic inequalities graphically → pages 51–53

Prior knowledge check

1 A = {factors of 12}
B = {factors of 20}
Write down the numbers in each of these sets:



a $A \cap B$ **b** $(A \cup B)'$

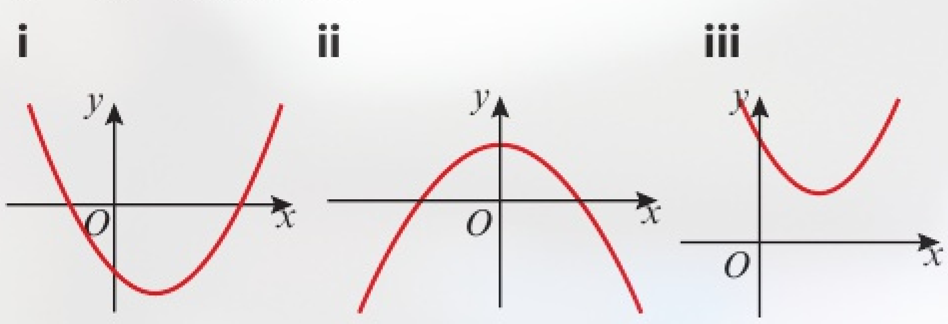
← International GCSE Mathematics

2 Simplify these expressions.

a $\sqrt{75}$ **b** $\dfrac{2\sqrt{45} + 3\sqrt{32}}{6}$ ← Section 1.5

3 Match the equations to the correct graph. Label the points of intersection with the axes and the coordinates of the turning point.

a $y = 9 - x^2$ **b** $y = (x - 2)^2 + 4$
c $y = (x - 7)(2x + 5)$



← Section 2.4

Food scientists use regions on graphs to optimise athletes' nutritional intake and ensure they satisfy the minimum dietary requirements for calories and vitamins.

3.1 Linear simultaneous equations

Linear simultaneous equations in two unknowns have one set of values that will make a pair of equations true at the same time.

The solution to this pair of simultaneous equations is $x = 5$, $y = 2$:

$x + 3y = 11$ (1) — $5 + 3(2) = 5 + 6 = 11$ ✓

$4x - 5y = 10$ (2) — $4(5) - 5(2) = 20 - 10 = 10$ ✓

- Linear simultaneous equations can be solved using **elimination** or substitution.

Example 1 SKILLS CRITICAL THINING

Solve the simultaneous equations:

a $2x + 3y = 8$
$3x - y = 23$

b $4x - 5y = 4$
$6x + 2y = 25$

a
$2x + 3y = 8$ (1)
$3x - y = 23$ (2) — First look for a way to eliminate x or y. Multiply equation (2) by 3 to get $3y$ in each equation.
$9x - 3y = 69$ (3) — Number this new equation (3).
$11x = 77$ — Then add equations (1) and (3), since the $3y$ terms have different signs and y will be eliminated.
$x = 7$

$14 + 3y = 8$ — Substitute $x = 7$ into equation (1) to find y.
$3y = 8 - 14$
$y = -2$
The solution is $x = 7$, $y = -2$. — Remember to check your solution by substituting into equation (2). $3(7) - (-2) = 21 + 2 = 23$ ✓ Note that you could also multiply equation (1) by 3 and equation (2) by 2 to get $6x$ in both equations. You could then subtract to eliminate x.

b
$4x - 5y = 4$ (1) — Multiply equation (1) by 3 and multiply equation (2) by 2 to get $12x$ in each equation.
$6x + 2y = 25$ (2)
$12x - 15y = 12$ (3)
$12x + 4y = 50$ (4)
$-19y = -38$ — Subtract, since the $12x$ terms have the same sign (both positive).
$y = 2$

$4x - 10 = 4$ — Substitute $y = 2$ into equation (1) to find x.
$4x = 14$
$x = 3\frac{1}{2}$
The solution is $x = 3\frac{1}{2}$, $y = 2$.

Example 2

Solve the simultaneous equations:

$2x - y = 1$
$4x + 2y = -30$

$2x - y = 1$ (1)
$4x + 2y = -30$ (2)

$y = 2x - 1$ — Rearrange an equation, in this case equation (1), to get either $x = \ldots$ or $y = \ldots$ (here $y = \ldots$).

$4x + 2(2x - 1) = -30$ — Substitute this into the other equation (here into equation (2) in place of y).

$4x + 4x - 2 = -30$
$8x = -28$
$x = -3\frac{1}{2}$ — Solve for x.

$y = 2(-3\frac{1}{2}) - 1 = -8$ — Substitute $x = -3\frac{1}{2}$ into equation (1) to find the value of y.

The solution is $x = -3\frac{1}{2}$, $y = -8$ — Remember to check your solution in equation (2). $4(-3.5) + 2(-8) = -14 - 16 = -30$ ✓

Exercise 3A

SKILLS PROBLEM SOLVING

1 Solve these simultaneous equations by elimination:

a $2x - y = 6$, $4x + 3y = 22$
b $7x + 3y = 16$, $2x + 9y = 29$
c $5x + 2y = 6$, $3x - 10y = 26$
d $2x - y = 12$, $6x + 2y = 21$
e $3x - 2y = -6$, $6x + 3y = 2$
f $3x + 8y = 33$, $6x = 3 + 5y$

2 Solve these simultaneous equations by substitution:

a $x + 3y = 11$, $4x - 7y = 6$
b $4x - 3y = 40$, $2x + y = 5$
c $3x - y = 7$, $10x + 3y = -2$
d $2y = 2x - 3$, $3y = x - 1$

3 Solve these simultaneous equations:

a $3x - 2y + 5 = 0$, $5(x + y) = 6(x + 1)$
b $\frac{x - 2y}{3} = 4$, $2x + 3y + 4 = 0$
c $3y = 5(x - 2)$, $3(x - 1) + y + 4 = 0$

Hint First rearrange both equations into the same form, e.g. $ax + by = c$.

E/P **4** $3x + ky = 8$
$x - 2ky = 5$
are simultaneous equations where k is a constant.

a Show that $x = 3$. **(3 marks)**
b Given that $y = \frac{1}{2}$, determine the value of k. **(1 mark)**

Problem-solving k is a constant, so it has the same value in both equations.

E/P **5** $2x - py = 5$
$4x + 5y + q = 0$
are simultaneous equations where p and q are constants.
The solution to this pair of simultaneous equations is $x = q$, $y = -1$.
Find the value of p and the value of q. **(5 marks)**

3.2 Quadratic simultaneous equations

You must be able to solve simultaneous equations where one equation is linear and one is quadratic.

To solve simultaneous equations involving one linear equation and one quadratic equation, you need to use a substitution method from the linear equation into the quadratic equation.

- Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

The solutions to this pair of simultaneous equations are $x = 4$, $y = -3$ and $x = 5.5$, $y = -1.5$:

$$x - y = 7 \quad (1)$$

$$y^2 + xy + 2x = 5 \quad (2)$$

$4 - (-3) = 7$ ✓ and $5.5 - (-1.5) = 7$ ✓

$(-3)^2 + (4)(-3) + 2(4) = 9 - 12 + 8 = 5$ ✓ and $(-1.5)^2 + (5.5)(-1.5) + 2(5.5) = 2.25 - 8.25 + 11 = 5$ ✓

Example 3

Solve the simultaneous equations:

$x + 2y = 3$
$x^2 + 3xy = 10$

$x + 2y = 3 \quad (1)$
$x^2 + 3xy = 10 \quad (2)$

The quadratic equation can contain terms involving y^2 and xy.

$x = 3 - 2y$

Rearrange linear equation (1) to get $x = \ldots$ or $y = \ldots$ (here $x = \ldots$).

$(3 - 2y)^2 + 3y(3 - 2y) = 10$

Substitute this into quadratic equation (2) (here in place of x).

$(3 - 2y)^2$ means $(3 - 2y)(3 - 2y)$ ← Section 1.2

$9 - 12y + 4y^2 + 9y - 6y^2 = 10$
$-2y^2 - 3y - 1 = 0$
$2y^2 + 3y + 1 = 0$
$(2y + 1)(y + 1) = 0$

Solve for y using factorisation.

$y = -\frac{1}{2}$ or $y = -1$

So $x = 4$ or $x = 5$

Find the corresponding x-values by substituting the y-values into linear equation (1), $x = 3 - 2y$.

Solutions are $x = 4$, $y = -\frac{1}{2}$ and $x = 5$, $y = -1$.

There are two solution pairs for x and y.

Exercise 3B

SKILLS PROBLEM SOLVING

1 Solve the simultaneous equations:

a $x + y = 11$, $xy = 30$
b $2x + y = 1$, $x^2 + y^2 = 1$
c $y = 3x$, $2y^2 - xy = 15$
d $3a + b = 8$, $3a^2 + b^2 = 28$
e $2u + v = 7$, $uv = 6$
f $3x + 2y = 7$, $x^2 + y = 8$

2 Solve the simultaneous equations:

a $2x + 2y = 7$, $x^2 - 4y^2 = 8$
b $x + y = 9$, $x^2 - 3xy + 2y^2 = 0$
c $5y - 4x = 1$, $x^2 - y^2 + 5x = 41$

3 Solve the simultaneous equations, giving your answers in their simplest surd form:

a $x - y = 6$
$xy = 4$

b $2x + 3y = 13$
$x^2 + y^2 = 78$

Watch out Use brackets when you are substituting an expression into an equation.

E/P 4 Solve the simultaneous equations:
$x + y = 3$
$x^2 - 3y = 1$ **(6 marks)**

E/P 5 a By eliminating y from the equations
$y = 2 - 4x$
$3x^2 + xy + 11 = 0$
show that $x^2 - 2x - 11 = 0$. **(2 marks)**

b Hence, or otherwise, solve the simultaneous equations
$y = 2 - 4x$
$3x^2 + xy + 11 = 0$
giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers. **(5 marks)**

P 6 One pair of solutions for the simultaneous equations
$y = kx - 5$
$4x^2 - xy = 6$
is $(1, p)$ where k and p are constants.

a Find the values of k and p.

b Find the second pair of solutions for the simultaneous equations.

Problem-solving If $(1, p)$ is a solution, then $x = 1$, $y = p$ satisfies both equations.

Challenge

$y - x = k$
$x^2 + y^2 = 4$
Given that the simultaneous equations have exactly one pair of solutions, show that
$k = \pm 2\sqrt{2}$

3.3 Simultaneous equations on graphs

You can represent the solutions of simultaneous equations graphically. As every point on a line or curve satisfies the equation of that line or curve, the points of **intersection** of two lines or curves satisfy both equations simultaneously.

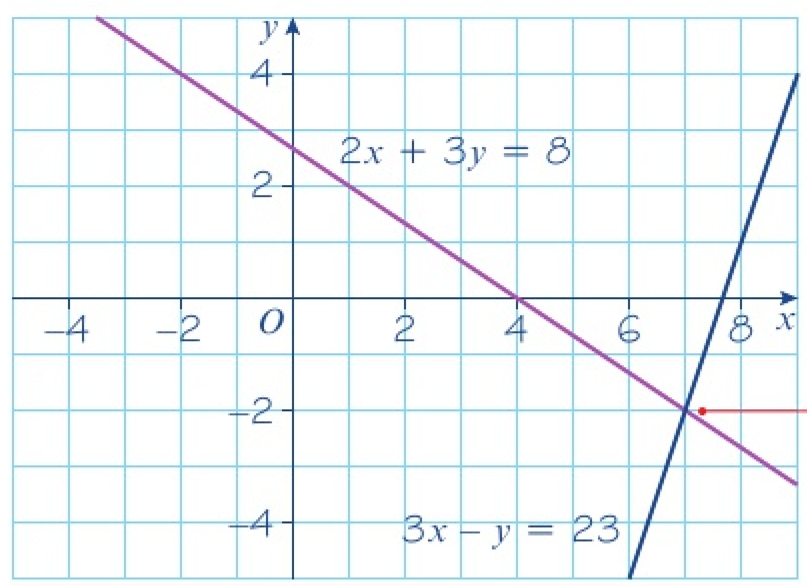
- Solutions to a pair of simultaneous equations represent the points of intersection of their graphs.

Example 4 SKILLS INTERPRETATION

a On the same axes, draw the graphs of:
$2x + 3y = 8$
$3x - y = 23$

b Use your graphs to write down the solutions to the simultaneous equations.

a



b The solution is (7, −2) or $x = 7$, $y = -2$.

Online Find the point of intersection graphically using technology.

The point of intersection is the solution to the simultaneous equations

$2x + 3y = 8$

$3x - y = 23$

This solution matches the **algebraic** solution to the simultaneous equations.

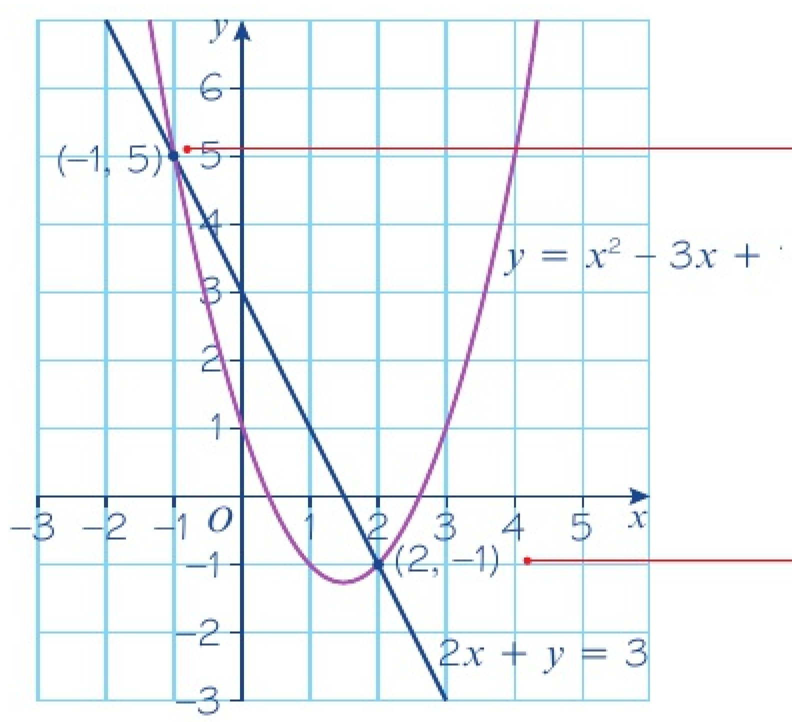
Example 5

a On the same axes, draw the graphs of:

$2x + y = 3$
$y = x^2 - 3x + 1$

b Use your graphs to write down the solutions to the simultaneous equations.

a



b The solutions are (−1, 5) or $x = -1$, $y = 5$ and (2, −1) or $x = 2$, $y = -1$.

There are *two* solutions. Each solution will have an x-value and a y-value.

Check your solutions by substituting into both equations.

$2(-1) + (5) = -2 + 5 = 3$ ✓ and
$5 = (-1)^2 - 3(-1) + 1 = 1 + 3 + 1 = 5$ ✓

$2(2) + (-1) = 4 - 1 = 3$ ✓ and
$-1 = (2)^2 - 3(2) + 1 = 4 - 6 + 1 = -1$ ✓

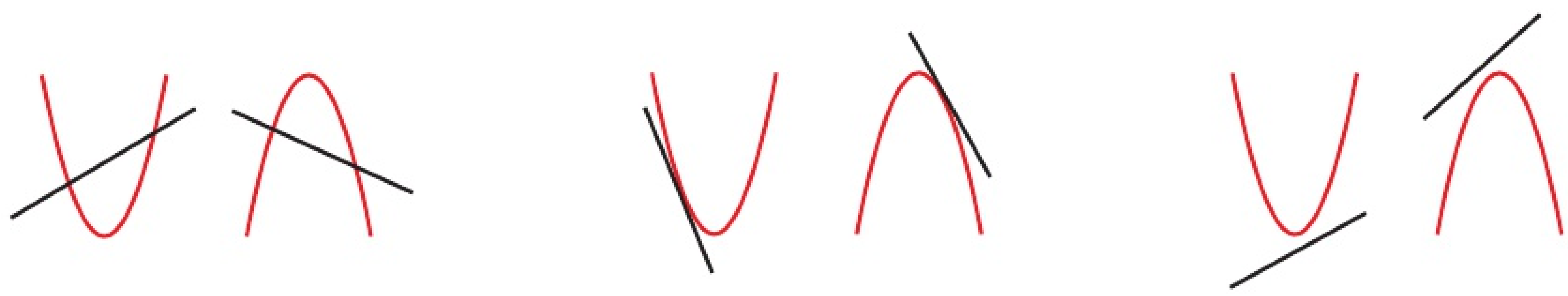
Online Plot the curve and the line using technology to find the two points of intersection.

The graph of a linear equation and the graph of a quadratic equation can either:

- intersect twice
- intersect once
- not intersect

After substituting, you can use the discriminant of the resulting quadratic equation to determine the number of points of intersection.

- For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:
  - $b^2 - 4ac > 0$ two real solutions
  - $b^2 - 4ac = 0$ one real solution
  - $b^2 - 4ac < 0$ no real solutions



Example 6

The line with equation $y = 2x + 1$ meets the curve with equation $kx^2 + 2y + (k - 2) = 0$ at exactly one point. Given that k is a positive constant

a find the value of k

b for this value of k, find the coordinates of the point of intersection.

Online Explore how the value of k affects the line and the curve using technology.

a
$$y = 2x + 1 \quad (1)$$
$$kx^2 + 2y + (k - 2) = 0 \quad (2)$$
$$kx^2 + 2(2x + 1) + (k - 2) = 0$$
$$kx^2 + 4x + 2 + k - 2 = 0$$
$$kx^2 + 4x + k = 0$$

Substitute $y = 2x + 1$ into equation (2) and simplify the quadratic equation. The resulting quadratic equation is in the form $ax^2 + bx + c = 0$ with $a = k$, $b = 4$ and $c = k$.

$$4^2 - 4 \times k \times k = 0$$

Problem-solving

You are told that the line meets the curve at exactly one point, so use the discriminant of the resulting quadratic. There will be exactly one solution, so $b^2 - 4ac = 0$.

$$16 - 4k^2 = 0$$
$$k^2 - 4 = 0$$
$$(k - 2)(k + 2) = 0$$

Factorise the quadratic to find the values of k.

$k = 2$ or $k = -2$

So $k = 2$

The solution is $k = +2$, as k is a positive constant.

b
$$2x^2 + 4x + 2 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$x = -1$$

Substitute $k = +2$ into the quadratic equation $kx^2 + 4x + k = 0$. Simplify and factorise to find the x-coordinate.

$$y = 2(-1) + 1 = -1$$

Substitute $x = -1$ into linear equation (1) to find the y-coordinate.

Point of intersection is $(-1, -1)$.

Check your answer by substituting into equation (2):
$$2x^2 + 2y = 0$$
$$2(-1)^2 + 2(-1) = 2 - 2 = 0 \checkmark$$

Exercise 3C SKILLS INTERPRETATION

1 In each case:
   i draw the graphs for each pair of equations on the same axes
   ii find the coordinates of the point of intersection.

   a $y = 3x - 5$
   $y = 3 - x$

   b $y = 2x - 7$
   $y = 8 - 3x$

   c $y = 3x + 2$
   $3x + y + 1 = 0$

2 a Use graph paper to accurately draw the graphs of $2y = 2x + 11$ and $y = 2x^2 - 3x - 5$ on the same axes.
   b Use your graphs to find the coordinates of the points of intersection.
   c Verify your solutions by substitution.

3 a On the same axes, sketch the curve with equation $x^2 + y = 9$ and the line with equation $2x + y = 6$.
   b Find the coordinates of the points of intersection.
   c Verify your solutions by substitution.

4 a On the same axes, sketch the curve with equation $y = (x - 2)^2$ and the line with equation $y = 3x - 2$.
   b Find the coordinates of the point of intersection.

Hint You need to use algebra in part **b** to find the coordinates.

5 Find the coordinates of the points at which the line with equation $y = x - 4$ intersects the curve with equation $y^2 = 2x^2 - 17$.

6 Find the coordinates of the points at which the line with equation $y = 3x - 1$ intersects the curve with equation $y^2 = xy + 15$.

(P) 7 Determine the number of points of intersection for these pairs of simultaneous equations.

   a $y = 6x^2 + 3x - 7$
   $y = 2x + 8$

   b $y = 4x^2 - 18x + 40$
   $y = 10x - 9$

   c $y = 3x^2 - 2x + 4$
   $7x + y + 3 = 0$

(E/P) 8 Given the simultaneous equations

$$2x - y = 1$$
$$x^2 + 4ky + 5k = 0$$

where k is a non-zero constant

   a show that $x^2 + 8kx + k = 0$. **(2 marks)**

   Given that $x^2 + 8kx + k = 0$ has equal roots

   b find the value of k **(3 marks)**
   c for this value of k, find the solution of the simultaneous equations. **(3 marks)**

3.4 Linear inequalities

You can solve linear inequalities using similar methods to those for solving linear equations.

- The solution of an inequality is the set of all real numbers x that make the inequality true.

Example 7 SKILLS PROBLEM SOLVING

Find the set of values of x for which:

a $5x + 9 \geqslant x + 20$ **b** $12 - 3x < 27$

c $3(x - 5) > 5 - 2(x - 8)$

Notation You can write the solution to this inequality using **set notation** as $\{x : x \geqslant 2.75\}$. This means the set of all values x for which x is greater than or equal to 2.75.

a $5x + 9 \geqslant x + 20$

$4x + 9 \geqslant 20$

$4x \geqslant 11$

$x \geqslant 2.75$ — Rearrange to get $x \geqslant \ldots$

b $12 - 3x < 27$ — Subtract 12 from both sides.

$-3x < 15$ — Divide both sides by −3. (You therefore need to turn round the inequality sign.)

$x > -5$

In set notation $\{x : x > -5\}$.

c $3(x - 5) > 5 - 2(x - 8)$

$3x - 15 > 5 - 2x + 16$ — Multiply out (note: $-2 \times -8 = +16$).

$5x > 5 + 16 + 15$ — Rearrange to get $x > \ldots$

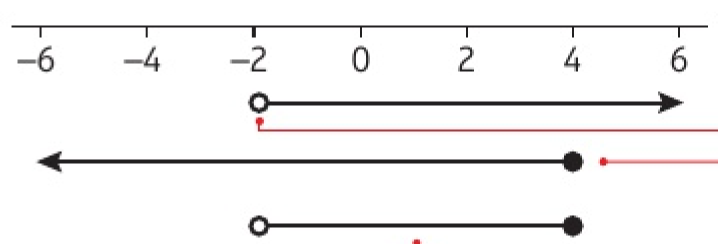
$5x > 36$

$x > 7.2$ — In set notation $\{x : x > 7.2\}$.

You may sometimes need to find the set of values for which **two** inequalities are true together. Number lines can be useful to find the solution.

Notation In set notation
$x > -2$ **and** $x \leqslant 4$ is written $\{x : -2 < x \leqslant 4\}$
or alternatively $\{x : x > -2\} \cap \{x : x \leqslant 4\}$
$x \leqslant -1$ **or** $x > 3$ is written $\{x : x \leqslant -1\} \cup \{x : x > 3\}$

For example, in the number line below the solution set is $x > -2$ **and** $x \leqslant 4$.

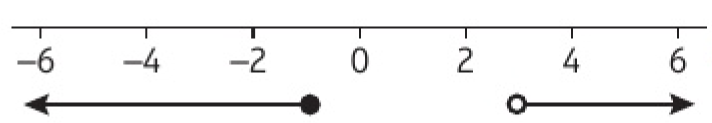


○ is used for < and > and means the end value is *not* included.

● is used for ⩽ and ⩾ and means the end value is included.

These are the only real values that satisfy both equalities simultaneously, so the solution is $-2 < x \leqslant 4$.

Here the solution sets are $x \leqslant -1$ **or** $x > 3$.



Here there is no **overlap** and the two inequalities have to be written separately as $x \leqslant -1$ or $x > 3$.

Example 8

Find the set of values of x for which:

a $3x - 5 < x + 8$ and $5x > x - 8$

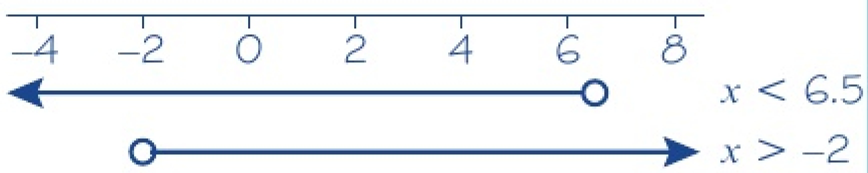
b $x - 5 > 1 - x$ or $15 - 3x > 5 + 2x$

a $3x - 5 < x + 8$ $\quad 5x > x - 8$

$2x - 5 < 8$ $\quad 4x > -8$

$2x < 13$ $\quad x > -2$

$x < 6.5$



So the required set of values is $-2 < x < 6.5$.

Draw a number line to **illustrate** the two inequalities.

The two sets of values overlap (intersect) where $-2 < x < 6.5$.

Notice here how this is written when x lies between two values.

In set notation this can be written as $\{x : -2 < x < 6.5\}$.

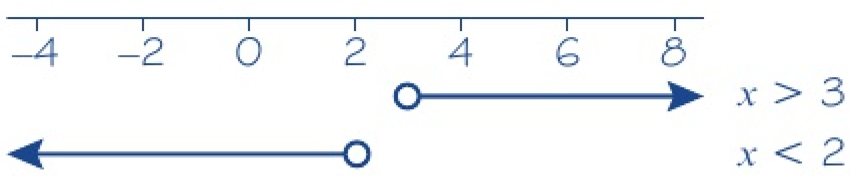
b $x - 5 > 1 - x$ $\quad 15 - 3x > 5 + 2x$

$2x - 5 > 1$ $\quad 10 - 3x > 2x$

$2x > 6$ $\quad 10 > 5x$

$x > 3$ $\quad 2 > x$

$x < 2$



The solution is $x > 3$ or $x < 2$.

Draw a number line. Note that there is no overlap between the two sets of values.

In set notation this can be written as $\{x : x < 2\} \cup \{x : x > 3\}$.

Exercise 3D

SKILLS REASONING/ARGUMENTATION

1 Find the set of values of x for which:

a $2x - 3 < 5$

b $5x + 4 \geqslant 39$

c $6x - 3 > 2x + 7$

d $5x + 6 \leqslant -12 - x$

e $15 - x > 4$

f $21 - 2x > 8 + 3x$

g $1 + x < 25 + 3x$

h $7x - 7 < 7 - 7x$

i $5 - 0.5x \geqslant 1$

j $5x + 4 > 12 - 2x$

2 Find the set of values of x for which:

a $2(x - 3) \geqslant 0$ **b** $8(1 - x) > x - 1$ **c** $3(x + 7) \leqslant 8 - x$

d $2(x - 3) - (x + 12) < 0$ **e** $1 + 11(2 - x) < 10(x - 4)$ **f** $2(x - 5) \geqslant 3(4 - x)$

g $12x - 3(x - 3) < 45$ **h** $x - 2(5 + 2x) < 11$ **i** $x(x - 4) \geqslant x^2 + 2$

j $x(5 - x) \geqslant 3 + x - x^2$ **k** $3x + 2x(x - 3) \leqslant 2(5 + x^2)$ **l** $x(2x - 5) \leqslant \frac{4x(x + 3)}{2} - 9$

3 Use set notation to describe the set of values of x for which:

a $3(x - 2) > x - 4$ and $4x + 12 > 2x + 17$

b $2x - 5 < x - 1$ and $7(x + 1) > 23 - x$

c $2x - 3 > 2$ and $3(x + 2) < 12 + x$

d $15 - x < 2(11 - x)$ and $5(3x - 1) > 12x + 19$

e $3x + 8 \leqslant 20$ and $2(3x - 7) \geqslant x + 6$

f $5x + 3 < 9$ or $5(2x + 1) > 27$

g $4(3x + 7) \leqslant 20$ or $2(3x - 5) \geqslant \frac{7 - 6x}{2}$

Challenge

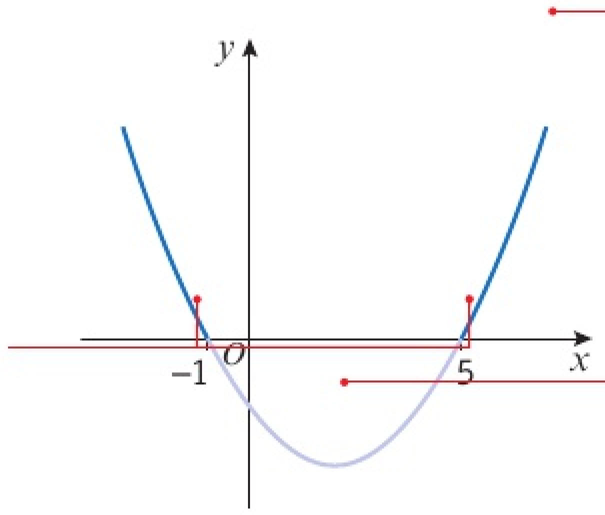
$A = \{x : 3x + 5 > 2\}$ $B = \left\{x : \frac{x}{2} + 1 \leqslant 3\right\}$ $C = \{x : 11 < 2x - 1\}$

Given that $A \cap (B \cup C) = \{x : p < x \leqslant q\} \cup \{x : x > r\}$, find the values of p, q and r.

3.5 Quadratic inequalities

- To solve a quadratic inequality:
  - **Rearrange so that the right-hand side of the inequality is 0**
  - **Solve the corresponding quadratic equation to find the critical values**
  - **Sketch the graph of the quadratic function**
  - **Use your sketch to find the required set of values.**

The sketch shows the graph of $f(x) = x^2 - 4x - 5$
$= (x + 1)(x - 5)$



The solutions to $f(x) = 0$ are $x = -1$ and $x = 5$. These are called the critical values.

The solutions to the quadratic inequality $x^2 - 4x - 5 > 0$ are the x-values when the curve is **above** the x-axis (the darker part of the curve). This is when $x < -1$ or $x > 5$. In set notation, the solution is $\{x : x < -1\} \cup \{x : x > 5\}$.

The solutions to the quadratic inequality $x^2 - 4x - 5 < 0$ are the x-values when the curve is **below** the x-axis (the lighter part of the curve). This is when $x > -1$ and $x < 5$ or $-1 < x < 5$. In set notation the solution is $\{x : -1 < x < 5\}$.

Example 9

Find the set of values of x for which:

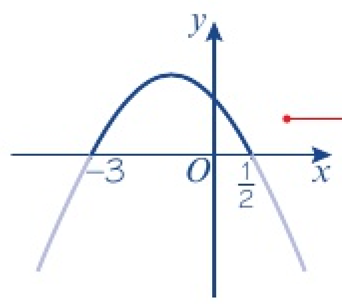
$3 - 5x - 2x^2 < 0$

$3 - 5x - 2x^2 = 0$ — Quadratic equation

$2x^2 + 5x - 3 = 0$ — Multiply by -1 (so it's easier to factorise).

$(2x - 1)(x + 3) = 0$

$x = \frac{1}{2}$ or $x = -3$ — $\frac{1}{2}$ and -3 are the critical values.



Draw a sketch to show the shape of the graph and the critical values.

Since the coefficient of x^2 is negative, the graph is 'upside-down ∪-shaped'. It crosses the x-axis at -3 and $\frac{1}{2}$. ← Section 2.4

So the required set of values is

$x < -3$ or $x > \frac{1}{2}$.

$3 - 5x - 2x^2 < 0$ ($y < 0$) for the outer parts of the graph, below the x-axis, as shown by the lighter parts of the curve.

In set notation this can be written as

$\{x : x < -3\} \cup \{x : x > \frac{1}{2}\}$.

Example 10

a Find the set of values of x for which $12 + 4x > x^2$.

b Hence find the set of values for which $12 + 4x > x^2$ and $5x - 3 > 2$.

a $12 + 4x > x^2$

$0 > x^2 - 4x - 12$

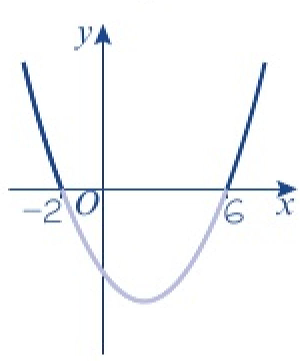
$x^2 - 4x - 12 < 0$

$x^2 - 4x - 12 = 0$

$(x + 2)(x - 6) = 0$

$x = -2$ or $x = 6$

Sketch $y = x^2 - 4x - 12$



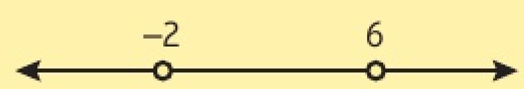
$x^2 - 4x - 12 < 0$

Solution: $-2 < x < 6$

You can use a table to check your solution.

$-2 < x < 6$

Use the critical values to split the real number line into sets.



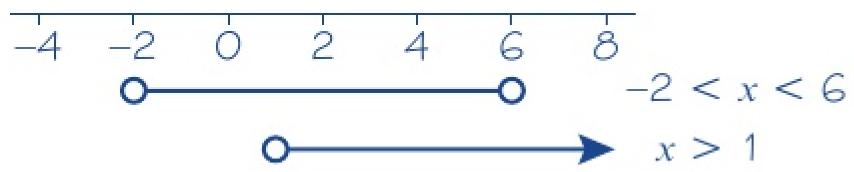
	$x < -2$	$-2 < x < 6$	$x > 6$
$x + 2$	−	+	+
$x - 6$	−	−	+
$(x + 2)(x - 6)$	+	−	+

For each set, check whether the set of values makes the value of the bracket positive or negative. For example, if $x < -2$, $(x + 2)$ is negative, $(x - 6)$ is negative, and $(x + 2)(x - 6)$ is (neg) × (neg) = positive.

In set notation the solution is $\{x : -2 < x < 6\}$.

b Solving $12 + 4x > x^2$ gives $-2 < x < 6$.

Solving $5x - 3 > 2$ gives $x > 1$.



The two sets of values overlap where $1 < x < 6$.

So the solution is $1 < x < 6$.

Problem-solving

This question is easier if you represent the information in more than one way. Use a sketch graph to solve the quadratic inequality, and use a number line to combine it with the linear inequality.

In set notation this can be written as $\{x : 1 < x < 6\}$.

Example 11 SKILLS INTERPRETATION

Find the set of values for which $\frac{6}{x} > 2$, $x \neq 0$

Watch out x could be either positive or negative, so you can't multiply both sides of this inequality by x. Instead, multiply both sides by x^2. Because x^2 is never negative, and $x \neq 0$ so $x^2 \neq 0$, the inequality sign stays the same.

$\frac{6}{x} > 2$

$6x > 2x^2$

$6x - 2x^2 > 0$

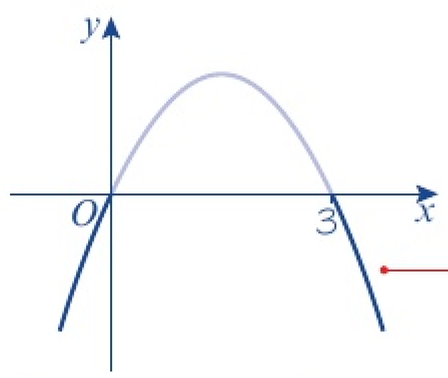
$6x - 2x^2 = 0$

Solve the corresponding quadratic equation to find the critical values.

$x(6 - 2x) = 0$

$x = 0$ or $x = 3$

$x = 0$ can still be a critical value even though $x \neq 0$. But it would *not* be part of the solution set, even if the inequality was $\geqslant$ rather than $>$.



Sketch $y = x(6 - 2x)$. You are interested in the values of x where the graph is above the x-axis.

The solution is $0 < x < 3$.

In set notation this can be written as $\{x : 0 < x < 3\}$.

Exercise 3E SKILLS INTERPRETATION

1 Find the set of values of x for which:

a $x^2 - 11x + 24 < 0$ **b** $12 - x - x^2 > 0$ **c** $x^2 - 3x - 10 > 0$

d $x^2 + 7x + 12 \geqslant 0$ **e** $7 + 13x - 2x^2 > 0$ **f** $10 + x - 2x^2 < 0$

g $4x^2 - 8x + 3 \leqslant 0$ **h** $-2 + 7x - 3x^2 < 0$ **i** $x^2 - 9 < 0$

j $6x^2 + 11x - 10 > 0$ **k** $x^2 - 5x > 0$ **l** $2x^2 + 3x \leqslant 0$

2 Find the set of values of x for which:

a $x^2 < 10 - 3x$ **b** $11 < x^2 + 10$

c $x(3 - 2x) > 1$ **d** $x(x + 11) < 3(1 - x^2)$

3 Use set notation to describe the set of values of x for which:

a $x^2 - 7x + 10 < 0$ and $3x + 5 < 17$
b $x^2 - x - 6 > 0$ and $10 - 2x < 5$
c $4x^2 - 3x - 1 < 0$ and $4(x + 2) < 15 - (x + 7)$
d $2x^2 - x - 1 < 0$ and $14 < 3x - 2$
e $x^2 - x - 12 > 0$ and $3x + 17 > 2$
f $x^2 - 2x - 3 < 0$ and $x^2 - 3x + 2 > 0$

(P) **4** Given that $x \neq 0$, find the set of values of x for which:

a $\frac{2}{x} < 1$
b $5 > \frac{4}{x}$
c $\frac{1}{x} + 3 > 2$
d $6 + \frac{5}{x} > \frac{8}{x}$
e $25 > \frac{1}{x^2}$
f $\frac{6}{x^2} + \frac{7}{x} \leqslant 3$

5 **a** Find the range of values of k for which the equation $x^2 - kx + (k + 3) = 0$ has no real roots.

b Find the range of values of p for which the roots of the equation $px^2 + px - 2 = 0$ are real.

Hint The quadratic equation $ax^2 + bx + c = 0$ has real roots if $b^2 - 4ac \geqslant 0$. ← Section 2.5

(E) **6** Find the set of values of x for which $x^2 - 5x - 14 > 0$. **(4 marks)**

(E) **7** Find the set of values of x for which

a $2(3x - 1) < 4 - 3x$ **(2 marks)**
b $2x^2 - 5x - 3 < 0$ **(4 marks)**
c **both** $2(3x - 1) < 4 - 3x$ **and** $2x^2 - 5x - 3 < 0$. **(2 marks)**

(E/P) **8** Given that $x \neq 3$, find the set of values for which $\frac{5}{x - 3} < 2$. **(6 marks)**

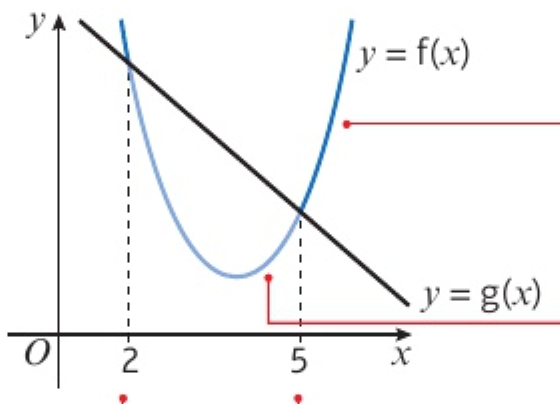
Problem-solving Multiply both sides of the inequality by $(x - 3)^2$.

(E/P) **9** The equation $kx^2 - 2kx + 3 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \leqslant k < 3$. **(4 marks)**

3.6 Inequalities on graphs

You may be asked to interpret graphically the solutions to inequalities by considering the graphs of functions that are related to them.

- The values of x for which the curve $y = f(x)$ is below the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.
- The values of x for which the curve $x = f(x)$ is above the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.



$f(x)$ is above $g(x)$ when $x < 2$ and when $x > 5$.
These values of x satisfy $f(x) > g(x)$.

$f(x)$ is below $g(x)$ when $2 < x < 5$.
These values of x satisfy $f(x) < g(x)$.

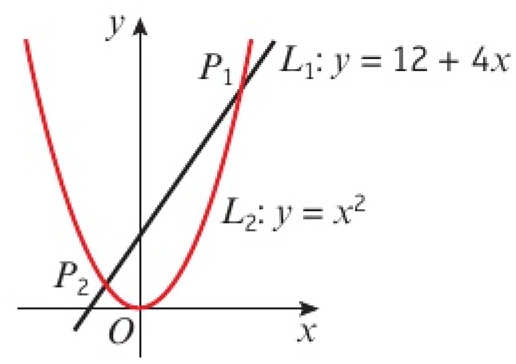
The solutions to $f(x) = g(x)$ are $x = 2$ and $x = 5$.

Example 12

L_1 has equation $y = 12 + 4x$.
L_2 has equation $y = x^2$.
The diagram shows a sketch of L_1 and L_2 on the same axes.

a Find the coordinates of P_1 and P_2, the points of intersection.

b Hence write down the solution to the inequality $12 + 4x > x^2$.



a $x^2 = 12 + 4x$
$x^2 - 4x - 12 = 0$

Equate to find the points of intersection, then rearrange to solve the quadratic equation.

$(x - 6)(x + 2) = 0$

Factorise to find the x-coordinates at the points of intersection.

$x = 6$ and $x = -2$
Substitute into $y = x^2$:
when $x = 6$, $y = 36$ P_1 (6, 36)
when $x = -2$, $y = 4$ P_2 (−2, 4)

b $12 + 4x > x^2$ when the graph of L_1 is above the graph of L_2
$-2 < x < 6$

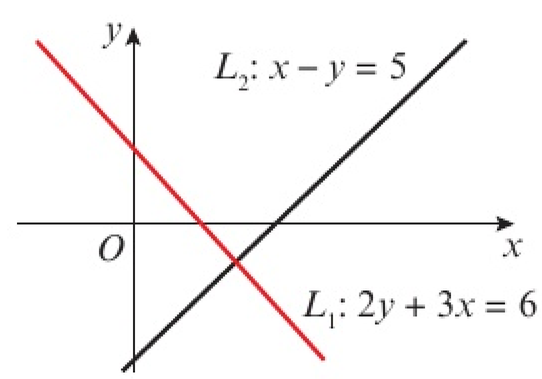
This is the range of values of x for which the graph of $y = 12 + 4x$ is above the graph of $y = x^2$, i.e. between the two points of intersection.
In set notation this is $\{x : -2 < x < 6\}$.

Exercise 3F SKILLS ANALYSIS

1 L_1 has equation $2y + 3x = 6$.
L_2 has equation $x - y = 5$.
The diagram shows a sketch of L_1 and L_2.

a Find the coordinates of P, the point of intersection.

b Hence write down the solution to the inequality $2y + 3x > x - y$.



2 For each pair of functions:

i Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes.

ii Find the coordinates of any points of intersection.

iii Write down the solutions to the inequality $f(x) \leqslant g(x)$.

a $f(x) = 3x - 7$, $g(x) = 13 - 2x$

b $f(x) = 8 - 5x$, $g(x) = 14 - 3x$

c $f(x) = x^2 + 5$, $g(x) = 5 - 2x$

d $f(x) = 3 - x^2$, $g(x) = 2x - 12$

e $f(x) = x^2 - 5$, $g(x) = 7x + 13$

f $f(x) = 7 - x^2$, $g(x) = 2x - 8$

(P) **3** Find the set of values of x for which the curve with equation $y = f(x)$ is below the line with equation $y = g(x)$.

a $f(x) = 3x^2 - 2x - 1$, $g(x) = x + 5$

b $f(x) = 2x^2 - 4x + 1$, $g(x) = 3x - 2$

c $f(x) = 5x - 2x^2 - 4$, $g(x) = -2x - 1$

d $f(x) = \frac{2}{x}, x \neq 0$, $g(x) = 1$

e $f(x) = \frac{3}{x^2} - \frac{4}{x}, x \neq 0$, $g(x) = -1$

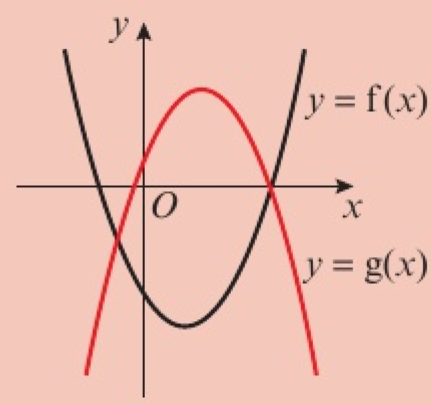
f $f(x) = \frac{2}{x+1}, x \neq -1$, $g(x) = 8$

Challenge

The sketch shows the graphs of

$f(x) = x^2 - 4x - 12$

$g(x) = 6 + 5x - x^2$



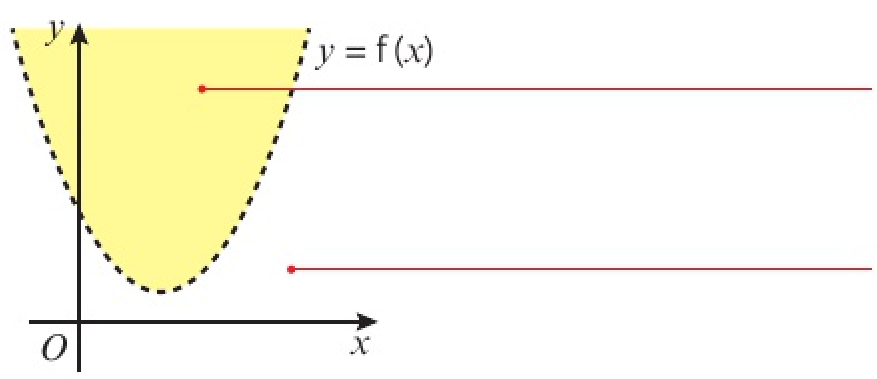
a Find the coordinates of the points of intersection.

b Find the set of values of x for which $f(x) < g(x)$. Give your answer in set notation.

3.7 Regions

You can use shading on graphs to identify **regions** that satisfy linear and quadratic inequalities.

- $y < f(x)$ represents the points on the **coordinate grid** below the curve $y = f(x)$.
- $y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.

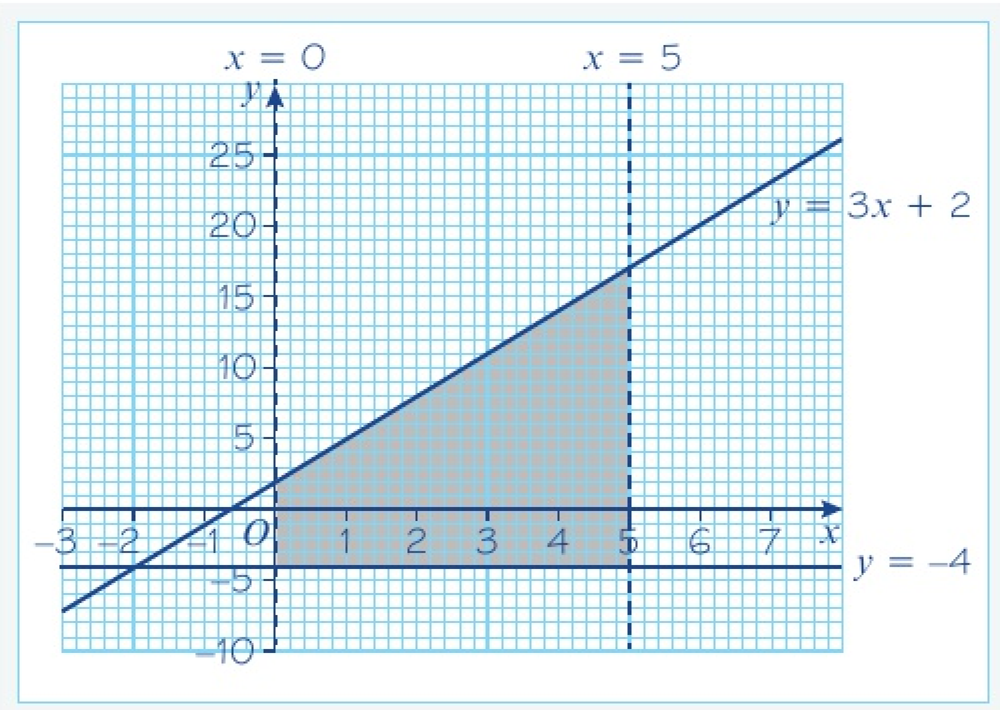


- If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is *not* included in the region and is represented by a dotted line.
- If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.

Example 13 SKILLS INTERPRETATION

On graph paper, shade the region that satisfies the inequalities:

$y \geqslant -4$, $x < 5$, $y \leqslant 3x + 2$ and $x > 0$



Draw dotted lines for $x = 0$, $x = 5$.

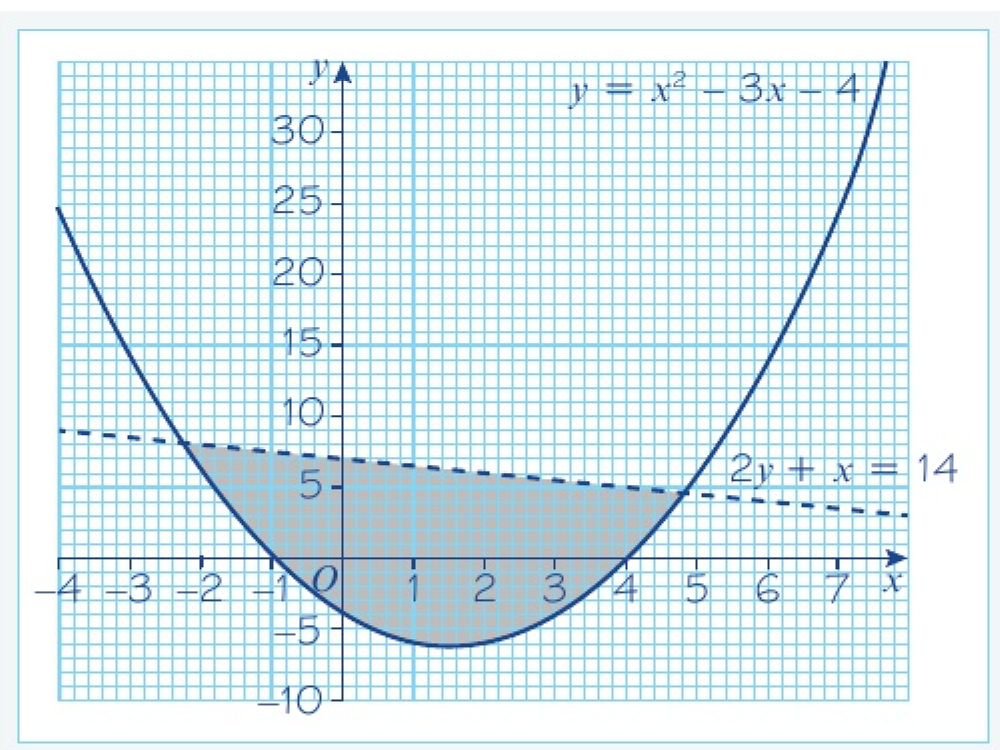
Shade the required region.
Test a point in the region. Try (1, 2).
For $x = 1$: $1 < 5$ and $1 > 0$ ✓
For $y = 2$: $2 \geqslant -2$ and $2 \leqslant 3 + 2$ ✓

Draw solid lines for $y = -4$, $y = 3x + 2$.

Example 14

On graph paper, shade the region that satisfies the inequalities:

$2y + x < 14$
$y \geqslant x^2 - 3x - 4$



Online Explore which regions on the graph satisfy which inequalities using technology.

Draw a dotted line for $2y + x = 14$ and a solid line for $y = x^2 - 3x - 4$.

Shade the required region.
Test a point in the region. Try (0, 0).
$0 + 0 < 14$ and $0 > 0 - 0 - 4$ ✓

Exercise 3G

SKILLS ANALYSIS

1 On a coordinate grid, shade the region that satisfies the inequalities:

$y > x - 2$, $y < 4x$ and $y \leqslant 5 - x$

2 On a coordinate grid, shade the region that satisfies the inequalities:

$x \geqslant -1$, $y + x < 4$, $2x + y \leqslant 5$ and $y > -2$

3 On a coordinate grid, shade the region that satisfies the inequalities:

$y < (3 - x)(2 + x)$ and $y + x \geqslant 3$

4 On a coordinate grid, shade the region that satisfies the inequalities:

$y > x^2 - 2$ and $y \leqslant 9 - x^2$

5 On a coordinate grid, shade the region that satisfies the inequalities:

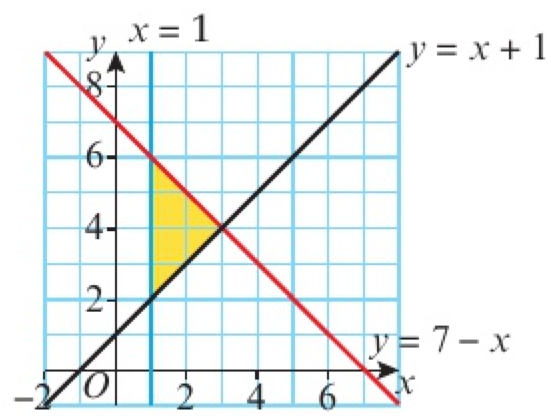
$y > (x - 3)^2$, $y + x \geqslant 5$ and $y < x - 1$

6 The sketch shows the graphs of the straight lines with equations:

$y = x + 1$, $y = 7 - x$ and $x = 1$

a Work out the coordinates of the points of intersection of the functions.

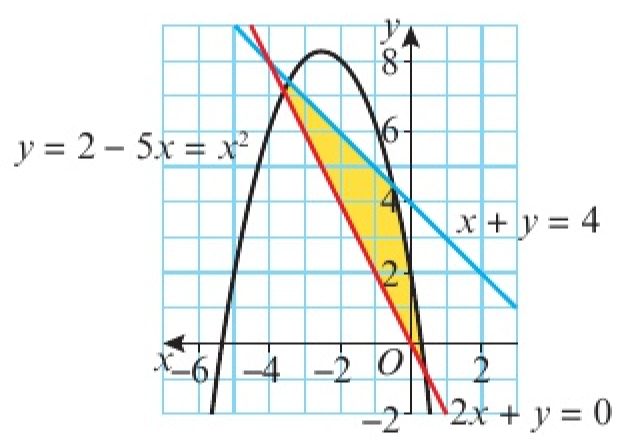
b Write down the set of inequalities that represent the shaded region shown in the sketch.



7 The sketch shows the graphs of the curves with equations:

$y = 2 - 5x - x^2$, $2x + y = 0$ and $x + y = 4$

Write down the set of inequalities that represent the shaded region shown in the sketch.



Ⓟ **8 a** On a coordinate grid, shade the region that satisfies the inequalities:

$y < x + 4$, $y + 5x + 3 \geqslant 0$, $y \geqslant -1$ and $x < 2$

b Work out the coordinates of the **vertices** of the shaded region.

c Which of the vertices lie within the region identified by the inequalities?

d Work out the area of the shaded region.

Problem-solving

A **vertex** is included only if both intersecting lines are included.

Chapter review 3

SKILLS EXECUTIVE FUNCTION

E **1** $2kx - y = 4$

$4kx + 3y = -2$

are two simultaneous equations, where k is a constant.

a Show that $y = -2$. **(3 marks)**

b Find an expression for x in terms of the constant k. **(1 mark)**

E **2** Solve the simultaneous equations:

$x + 2y = 3$

$x^2 - 4y^2 = -33$ **(7 marks)**

E **3** Given the simultaneous equations

$x - 2y = 1$

$3xy - y^2 = 8$

a Show that $5y^2 + 3y - 8 = 0$. **(2 marks)**

b Hence find the pairs (x, y) for which the simultaneous equations are satisfied. **(5 marks)**

E **4** **a** By eliminating y from the equations

$x + y = 2$

$x^2 + xy - y^2 = -1$

show that $x^2 - 6x + 3 = 0$. **(2 marks)**

b Hence, or otherwise solve the simultaneous equations

$x + y = 2$

$x^2 + xy - y^2 = -1$

giving x and y in the form $a \pm b\sqrt{6}$, where a and b are integers. **(5 marks)**

E **5** **a** Given that $3^x = 9^{y-1}$, show that $x = 2y - 2$. **(1 mark)**

b Solve the simultaneous equations:

$x = 2y - 2$

$x^2 = y^2 + 7$ **(6 marks)**

E **6** Solve the simultaneous equations:

$x + 2y = 3$

$x^2 - 2y + 4y^2 = 18$ **(7 marks)**

E/P **7** The curve and the line given by the equations

$kx^2 - xy + (k + 1)x = 1$

$-\frac{k}{2}x + y = 1$

where k is a non-zero constant, intersect at a single point.

a Find the value of k. **(5 marks)**

b Give the coordinates of the point of intersection of the line and the curve. **(3 marks)**

(E) **8** Give your answers in set notation.

a Solve the inequality $3x - 8 > x + 13$. **(2 marks)**

b Solve the inequality $x^2 - 5x - 14 > 0$. **(4 marks)**

(E) **9** Find the set of values of x for which $(x - 1)(x - 4) < 2(x - 4)$. **(6 marks)**

(E) **10** **a** Use algebra to solve $(x - 1)(x + 2) = 18$. **(2 marks)**

b Hence, or otherwise, find the set of values of x for which $(x - 1)(x + 2) > 18$. Give your answer in set notation. **(2 marks)**

11 Find the set of values of x for which:

a $6x - 7 < 2x + 3$ **(2 marks)**

b $2x^2 - 11x + 5 < 0$ **(4 marks)**

c $5 < \frac{20}{x}$ **(4 marks)**

d both $6x - 7 < 2x + 3$ and $2x^2 - 11x + 5 < 0$. **(2 marks)**

(E) **12** Find the set of values of x that satisfy $\frac{8}{x^2} + 1 \leqslant \frac{9}{x}$, $x \neq 0$ **(5 marks)**

(E) **13** Find the values of k for which $kx^2 + 8x + 5 = 0$ has real roots. **(3 marks)**

(E/P) **14** The equation $2x^2 + 4kx - 5k = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $-\frac{5}{2} < k < 0$. **(3 marks)**

(E) **15** **a** Sketch the graphs of $y = f(x) = x^2 + 2x - 15$ and $g(x) = 6 - 2x$ on the same axes. **(4 marks)**

b Find the coordinates of any points of intersection. **(3 marks)**

c Write down the set of values of x for which $f(x) > g(x)$. **(1 mark)**

(E) **16** Find the set of values of x for which the curve with equation $y = 2x^2 + 3x - 15$ is below the line with equation $y = 8 + 2x$. **(5 marks)**

(E) **17** On a coordinate grid, shade the region that satisfies the inequalities:

$y > x^2 + 4x - 12$ and $y < 4 - x^2$ **(5 marks)**

(E/P) **18** **a** On a coordinate grid, shade the region that satisfies the inequalities

$y + x < 6$, $y < 2x + 9$, $y > 3$ and $x > 0$ **(6 marks)**

b Work out the area of the shaded region. **(2 marks)**

Challenge

1 Find the possible values of k for the quadratic equation $2kx^2 + 5kx + 5k - 3 = 0$ to have real roots.

2 A straight line has equation $y = 2x - k$ and a parabola has equation $y = 3x^2 + 2kx + 5$ where k is a constant. Find the range of values of k for which the line and the parabola do *not* intersect.

Summary of key points

1 Linear simultaneous equations can be solved using elimination or substitution.

2 Simultaneous equations with one linear and one quadratic equation can have up to two pairs of solutions. You need to make sure the solutions are paired correctly.

3 The solutions of a pair of simultaneous equations represent the points of intersection of their graphs.

4 For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:

- $b^2 - 4ac > 0$ two real solutions
- $b^2 - 4ac = 0$ one real solution
- $b^2 - 4ac < 0$ no real solutions

5 The solution of an inequality is the set of all real numbers x that make the inequality true.

6 To solve a quadratic inequality:

- Rearrange so that the right-hand side of the inequality is 0
- Solve the corresponding quadratic equation to find the critical values
- Sketch the graph of the quadratic function
- Use your sketch to find the required set of values.

7 The values of x for which the curve $y = f(x)$ is below the curve $y = g(x)$ satisfy the inequality $f(x) < g(x)$.

The values of x for which the curve $y = f(x)$ is above the curve $y = g(x)$ satisfy the inequality $f(x) > g(x)$.

8 $y < f(x)$ represents the points on the coordinate grid below the curve $y = f(x)$.

$y > f(x)$ represents the points on the coordinate grid above the curve $y = f(x)$.

9 If $y > f(x)$ or $y < f(x)$ then the curve $y = f(x)$ is *not* included in the region and is represented by a dotted line.

If $y \geqslant f(x)$ or $y \leqslant f(x)$ then the curve $y = f(x)$ is included in the region and is represented by a solid line.

4 GRAPHS AND TRANSFORMATIONS

1.11
1.12

Learning objectives

After completing this chapter you should be able to:

- Sketch cubic graphs → pages 58–61
- Sketch reciprocal graphs of the form $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ → pages 62–63
- Use intersection points of graphs to solve equations → pages 63–66
- Translate graphs → pages 67–71
- Stretch graphs → pages 71–74
- Transform graphs of unfamiliar functions → pages 75–77

Prior knowledge check

1 Factorise these quadratic expressions:
a $x^2 + 6x + 5$ **b** $x^2 - 4x + 3$

← International GCSE Mathematics

2 Sketch the graphs of the following functions:
a $y = (x + 2)(x - 3)$ **b** $y = x^2 - 6x - 7$ ← Section 2.4

3 **a** Copy and complete the table of values for the function $y = x^3 + x - 2$.

x	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
y	−12	−6.875			−2	−1.375			

b Use your table of values to draw the graph of $y = x^3 + x - 2$. ← International GCSE Mathematics

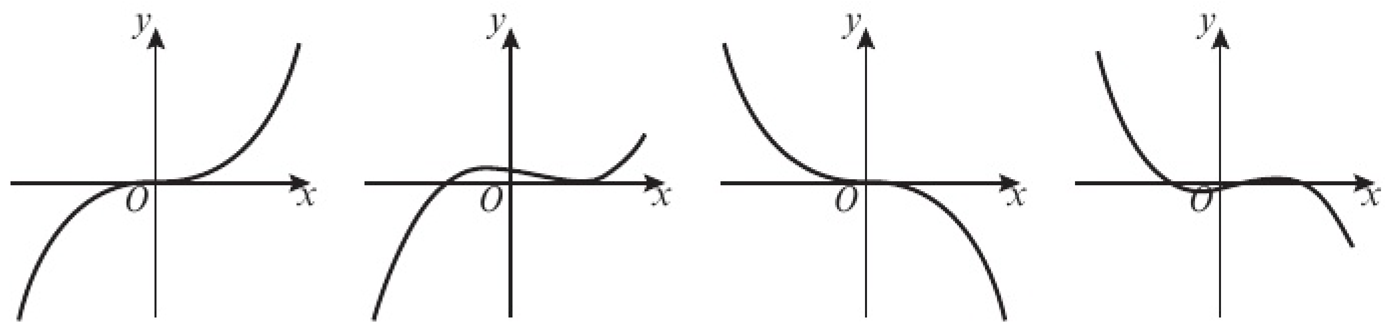
4 Solve each pair of simultaneous equations:
a $y = 2x$
$x + y = 6$
b $y = x^2$
$y = 2x - 1$ ← Sections 3.1, 3.2

Many complicated functions can be understood by transforming simpler functions using stretches, reflections and translations. Particle physicists compare observed results with transformations of known functions to determine the nature of subatomic particles.

4.1 Cubic graphs

A **cubic** function has the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers and a is non-zero.

The graph of a cubic function can take several different forms, depending on the exact nature of the function.



For these two functions a is positive. For these two functions a is negative.

- If p is a root of the function f(x), then the graph of y = f(x) touches or crosses the x-axis at the point (p, 0).

You can sketch the graph of a cubic function by finding the roots of the function.

Example 1

Sketch the curves with the following equations and show the points where they cross the coordinate axes.

a $y = (x - 2)(1 - x)(1 + x)$ **b** $y = x(x + 1)(x + 2)$

Online Explore the graph of $y = (x - p)(x - q)(x - r)$ where p, q and r are constants using technology.

a $y = (x - 2)(1 - x)(1 + x)$

$0 = (x - 2)(1 - x)(1 + x)$ — Put $y = 0$ and solve for x.

So $x = 2$, $x = 1$ or $x = -1$

So the curve crosses the x-axis at (2, 0), (1, 0) and (−1, 0).

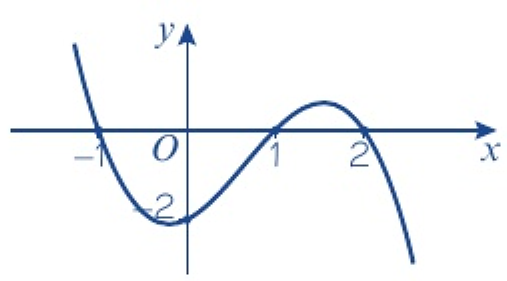
When $x = 0$, $y = -2 \times 1 \times 1 = -2$ — Find the value of y when $x = 0$.

So the curve crosses the y-axis at (0, −2).

$x \to \infty$, $y \to -\infty$

$x \to -\infty$, $y \to \infty$ — Check what happens to y for large positive and negative values of x.

The x^3 term in the expanded function would be $x \times (-x) \times x = -x^3$ so the curve has a negative x^3 coefficient.



b $y = x(x + 1)(x + 2)$

$0 = x(x + 1)(x + 2)$ — Put $y = 0$ and solve for x.

So $x = 0$, $x = -1$ or $x = -2$

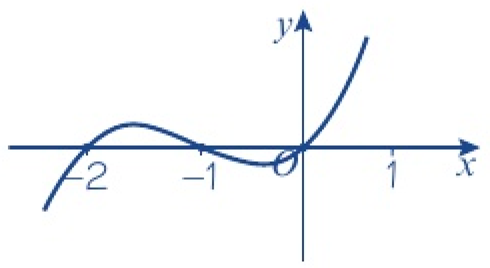
So the curve crosses the x-axis at (0, 0), (−1, 0) and (−2, 0).

> You know that the curve crosses the x-axis at (0, 0) so you don't need to calculate the y-intercept separately.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

> Check what happens to y for large positive and negative values of x.



> The x^3 term in the expanded function would be $x \times x \times x = x^3$ so the curve has a positive x^3 coefficient.

Example 2

Sketch the following curves:

a $y = (x - 1)^2(x + 1)$ **b** $y = x^3 - 2x^2 - 3x$ **c** $y = (x - 2)^3$

a $y = (x - 1)^2(x + 1)$

$0 = (x - 1)^2(x + 1)$

> Put $y = 0$ and solve for x.

So $x = 1$ or $x = -1$

So the curve crosses the x-axis at (−1, 0) and touches the x-axis at (1, 0).

> $(x - 1)$ is squared so $x = 1$ is a 'double' repeated root. This means that the curve just touches the x-axis at (1, 0).

When $x = 0$, $y = (-1)^2 \times 1 = 1$

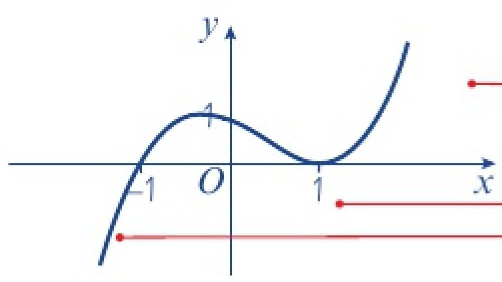
> Find the value of y when $x = 0$.

So the curve crosses the y-axis at (0, 1).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

> Check what happens to y for large positive and negative values of x.



> $x \to \infty, y \to \infty$

> $x = 1$ is a 'double' repeated root.

> $x \to -\infty, y \to -\infty$

b $y = x^3 - 2x^2 - 3x$

$= x(x^2 - 2x - 3)$

$= x(x - 3)(x + 1)$

> First factorise.

$0 = x(x - 3)(x + 1)$

> Put $y = 0$ and solve for x.

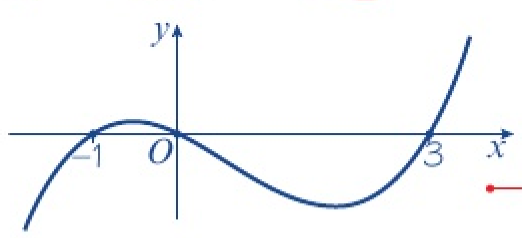
So $x = 0$, $x = 3$ or $x = -1$

So the curve crosses the x-axis at (0, 0), (3, 0) and (−1, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

> Check what happens to y for large positive and negative values of x.



> This is a cubic curve with a positive coefficient of x^3 and three distinct roots.

c $y = (x-2)^3$

$0 = (x-2)^3$

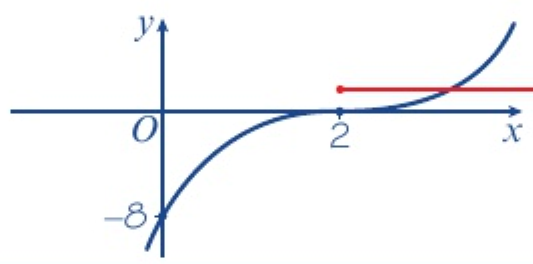
So $x = 2$ and the curve crosses the x-axis at (2, 0) only.

When $x = 0$, $y = (-2)^3 = -8$

So the curve crosses the y-axis at (0, −8).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



Put $y = 0$ and solve for x.

Check what happens to y for large positive and negative values of x.

$x = 2$ is a 'triple' repeated root.

Example 3

Sketch the curve with equation $y = (x-1)(x^2+x+2)$.

$y = (x-1)(x^2+x+2)$

$0 = (x-1)(x^2+x+2)$

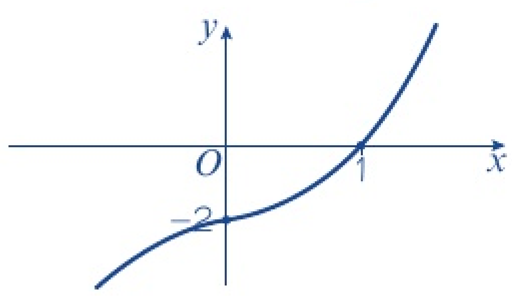
So $x = 1$ only and the curve crosses the x-axis at (1, 0).

When $x = 0$, $y = (-1)(2) = -2$

So the curve crosses the y-axis at (0, −2).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

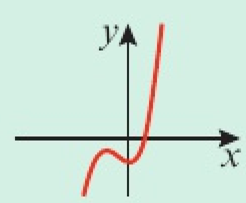


The quadratic factor $x^2 + x + 2$ gives no solutions since the discriminant $b^2 - 4ac = (1)^2 - 4(1)(2) = -7$.
← Section 2.5

Watch out A cubic graph could intersect the x-axis at 1, 2 or 3 points.

Check what happens to y for large positive and negative values of x.

Hint You haven't got enough information to know the exact shape of the graph. It could also be shaped like this:



Exercise 4A

SKILLS ANALYSIS

1 Sketch the following curves and indicate clearly the points of intersection with the axes:

a $y = (x-3)(x-2)(x+1)$ **b** $y = (x-1)(x+2)(x+3)$

c $y = (x+1)(x+2)(x+3)$ **d** $y = (x+1)(1-x)(x+3)$

e $y = (x-2)(x-3)(4-x)$ **f** $y = x(x-2)(x+1)$

g $y = x(x+1)(x-1)$ **h** $y = x(x+1)(1-x)$

i $y = (x-2)(2x-1)(2x+1)$ **j** $y = x(2x-1)(x+3)$

2 Sketch the curves with the following equations:

a $y = (x + 1)^2(x - 1)$ **b** $y = (x + 2)(x - 1)^2$ **c** $y = (2 - x)(x + 1)^2$
d $y = (x - 2)(x + 1)^2$ **e** $y = x^2(x + 2)$ **f** $y = (x - 1)^2x$
g $y = (1 - x)^2(3 + x)$ **h** $y = (x - 1)^2(3 - x)$ **i** $y = x^2(2 - x)$
j $y = x^2(x - 2)$

3 Factorise the following equations and then sketch the curves:

a $y = x^3 + x^2 - 2x$ **b** $y = x^3 + 5x^2 + 4x$ **c** $y = x^3 + 2x^2 + x$
d $y = 3x + 2x^2 - x^3$ **e** $y = x^3 - x^2$ **f** $y = x - x^3$
g $y = 12x^3 - 3x$ **h** $y = x^3 - x^2 - 2x$ **i** $y = x^3 - 9x$
j $y = x^3 - 9x^2$

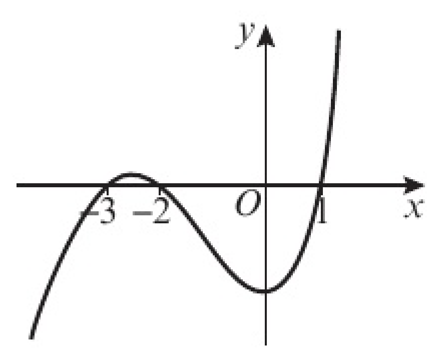
4 Sketch the following curves and indicate the coordinates of the points where the curves cross the axes:

a $y = (x - 2)^3$ **b** $y = (2 - x)^3$ **c** $y = (x - 1)^3$ **d** $y = (x + 2)^3$
e $y = -(x + 2)^3$ **f** $y = (x + 3)^3$ **g** $y = (x - 3)^3$ **h** $y = (1 - x)^3$
i $y = -(x - 2)^3$ **j** $y = -\left(x - \frac{1}{2}\right)^3$

E/P **5** The graph of $y = x^3 + bx^2 + cx + d$ is shown opposite, where b, c and d are real constants.

a Find the values of b, c and d. **(3 marks)**

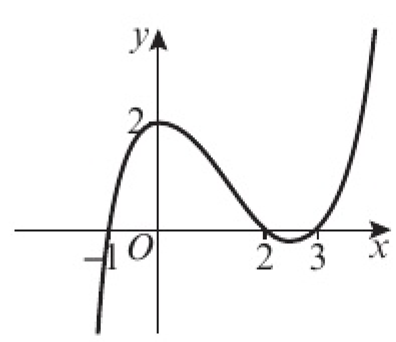
b Write down the coordinates of the point where the curve crosses the y-axis. **(1 mark)**



Problem-solving

Start by writing the equation in the form $y = (x - p)(x - q)(x - r)$.

E/P **6** The graph of $y = ax^3 + bx^2 + cx + d$ is shown opposite, where a, b, c and d are real constants.
Find the values of a, b, c and d. **(4 marks)**



E **7** Given that $f(x) = (x - 10)(x^2 - 2x) + 12x$

a Express $f(x)$ in the form $x(ax^2 + bx + c)$ where a, b and c are real constants. **(3 marks)**

b Hence factorise $f(x)$ completely. **(2 marks)**

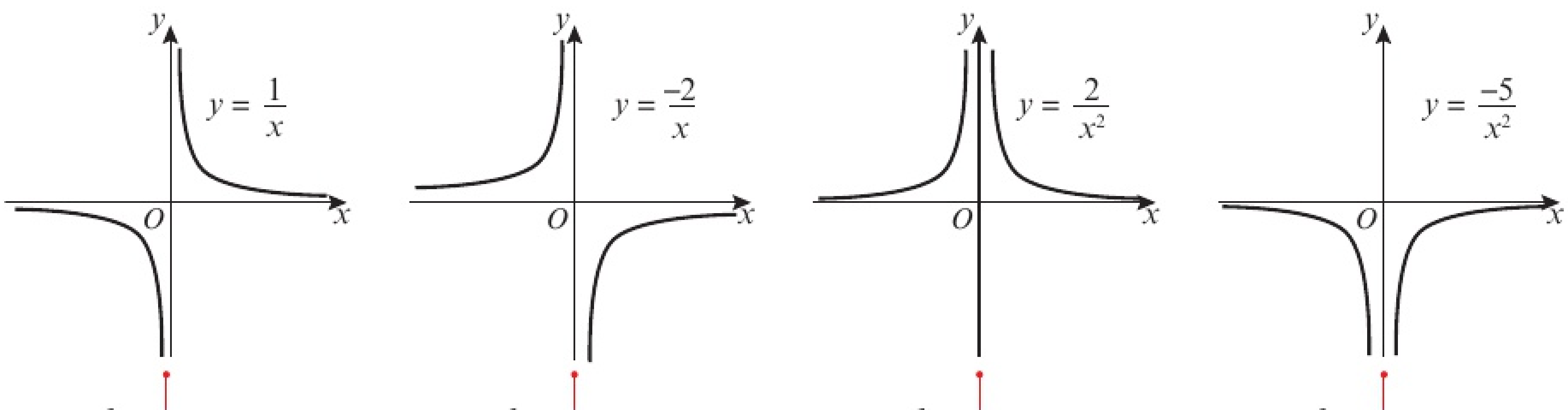
c Sketch the graph of $y = f(x)$ showing clearly the points where the graph intersects the axes. **(3 marks)**

4.2 Reciprocal graphs

You can sketch graphs of **reciprocal functions** such as $y = \frac{1}{x}$, $y = \frac{1}{x^2}$ and $y = -\frac{2}{x}$ by considering their asymptotes.

- The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.

Notation An **asymptote** is a line which the graph approaches but never reaches.

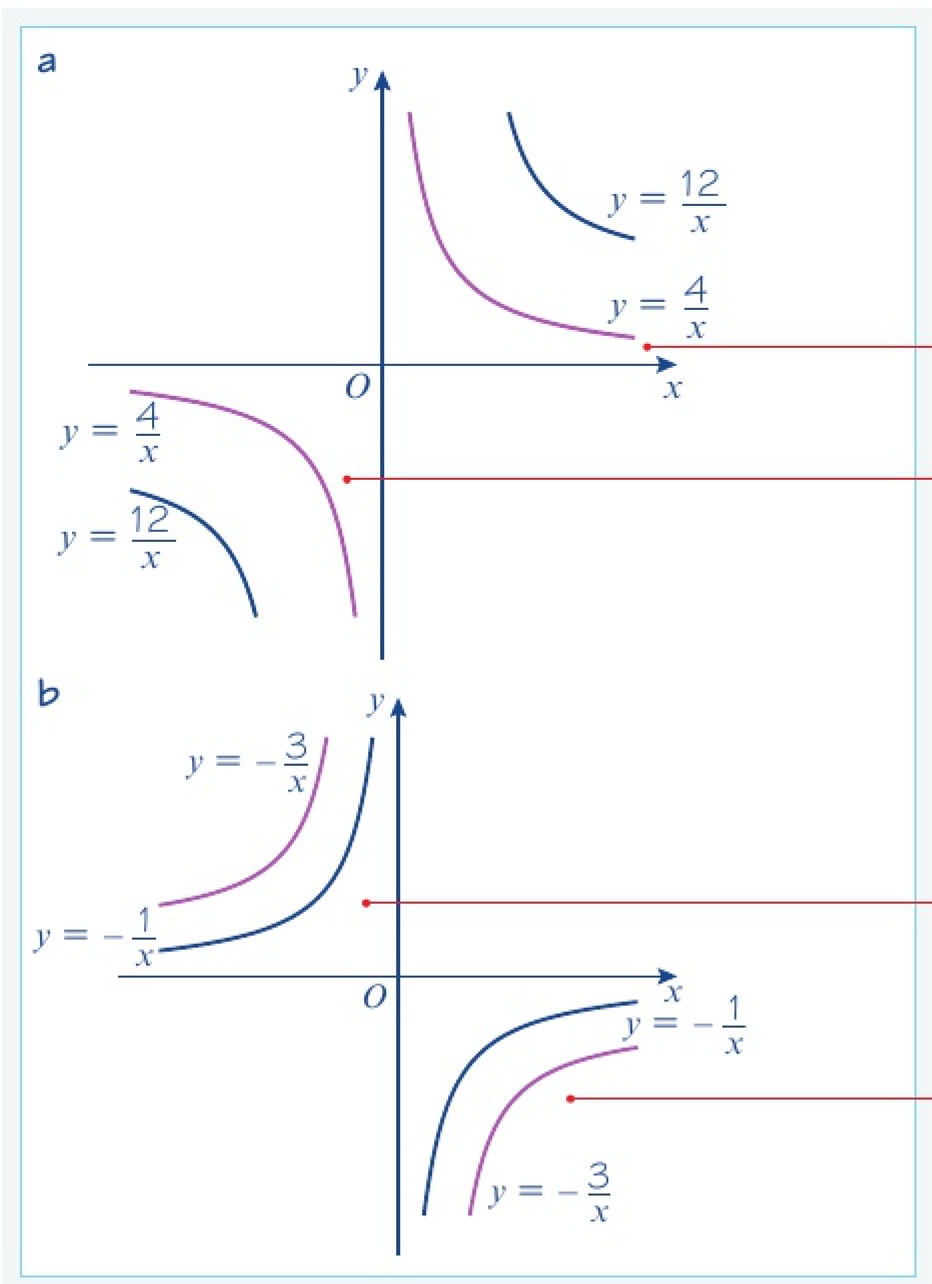


$y = \frac{k}{x}$ with $k > 0$ | $y = \frac{k}{x}$ with $k < 0$ | $y = \frac{k}{x^2}$ with $k > 0$ | $y = \frac{k}{x^2}$ with $k < 0$

Example 4 SKILLS INTERPRETATION

Sketch on the same diagram:

a $y = \frac{4}{x}$ and $y = \frac{12}{x}$ **b** $y = -\frac{1}{x}$ and $y = -\frac{3}{x}$ **c** $y = \frac{4}{x^2}$ and $y = \frac{10}{x^2}$



Online Explore the graph of $y = \frac{a}{x}$ for different values of a using technology.

This is a $y = \frac{k}{x}$ graph with $k > 0$

In this **quadrant**, $x > 0$

so for any values of x: $\frac{12}{x} > \frac{4}{x}$

In this quadrant, $x < 0$

so for any values of x: $\frac{12}{x} < \frac{4}{x}$

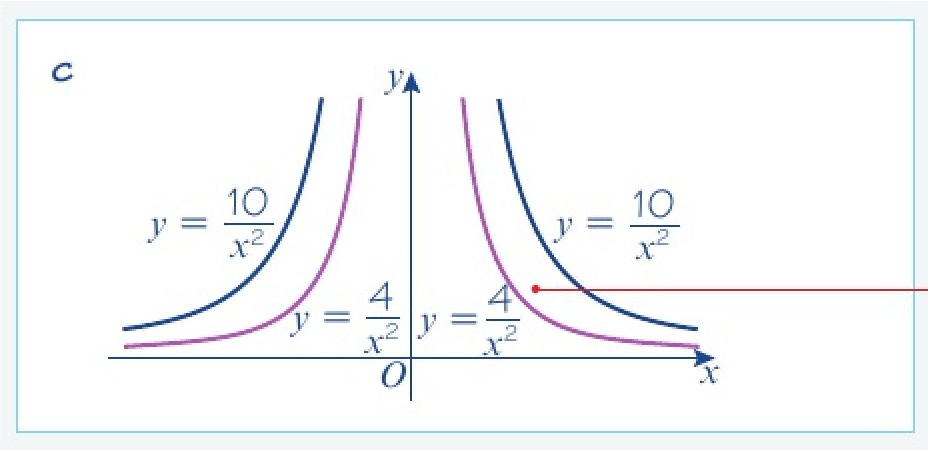
This is a $y = \frac{k}{x}$ graph with $k < 0$

In this quadrant, $x < 0$

so for any values of x: $-\frac{3}{x} > -\frac{1}{x}$

In this quadrant, $x > 0$

so for any values of x: $-\frac{3}{x} < -\frac{1}{x}$



This is a $y = \frac{k}{x^2}$ graph with $k > 0$.
x^2 is always positive and $k > 0$ so the y-values are all positive.

Exercise 4B SKILLS INTERPRETATION

1 Use a separate diagram to sketch each pair of graphs:

a $y = \frac{2}{x}$ and $y = \frac{4}{x}$

b $y = \frac{2}{x}$ and $y = -\frac{2}{x}$

c $y = -\frac{4}{x}$ and $y = -\frac{2}{x}$

d $y = \frac{3}{x}$ and $y = \frac{8}{x}$

e $y = -\frac{3}{x}$ and $y = -\frac{8}{x}$

2 Use a separate diagram to sketch each pair of graphs:

a $y = \frac{2}{x^2}$ and $y = \frac{5}{x^2}$

b $y = \frac{3}{x^2}$ and $y = -\frac{3}{x^2}$

c $y = -\frac{2}{x^2}$ and $y = -\frac{6}{x^2}$

4.3 Points of intersection

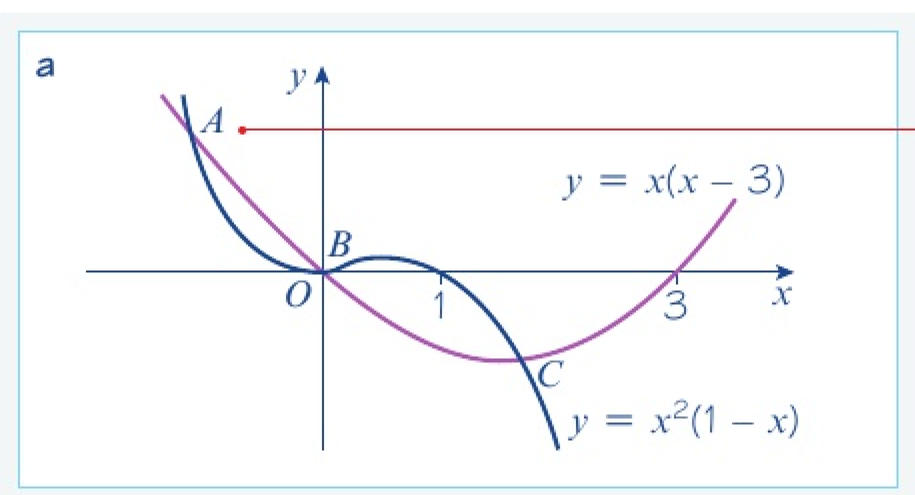
You can sketch curves of functions to show points of intersection and solutions to equations.

- The x-coordinate(s) at the **point(s)** of intersection of the curves with equations $y = f(x)$ and $y = g(x)$ are the solution(s) to the equation $f(x) = g(x)$.

Example 5

a On the same diagram, sketch the curves with equations $y = x(x - 3)$ and $y = x^2(1 - x)$.

b Find the coordinates of the points of intersection.



A cubic curve will eventually get **steeper** than a quadratic curve, so the graphs will intersect for some negative value of x.

b From the graph there are three points where the curves cross, labelled A, B and C. The x-coordinates are given by the solutions to the equation.

$$x(x - 3) = x^2(1 - x)$$

$$x^2 - 3x = x^2 - x^3$$

$$x^3 - 3x = 0$$

$$x(x^2 - 3) = 0$$

So $x = 0$ or $x^2 = 3$

So $x = -\sqrt{3}, 0, \sqrt{3}$

Substitute into $y = x^2(1 - x)$

The points of intersection are:

$A(-\sqrt{3}, 3 + 3\sqrt{3})$

$B(0, 0)$

$C(\sqrt{3}, 3 - 3\sqrt{3})$

There are three points of intersection, so the equation $x(x - 3) = x^2(1 - x)$ has three real roots.

Multiply out brackets.
Collect terms on one side.
Factorise.

The graphs intersect for these values of x, so you can substitute into either equation to find the y-coordinates.

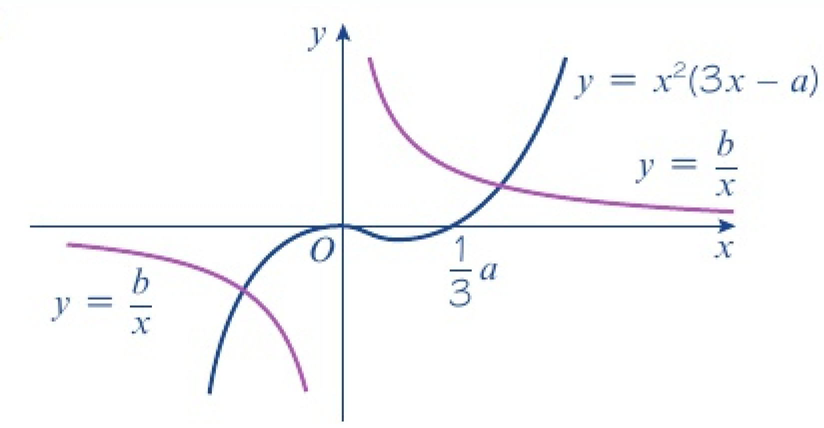
Leave your answers in surd form.

Example 6 SKILLS PROBLEM SOLVING

a On the same diagram, sketch the curves with equations $y = x^2(3x - a)$ and $y = \frac{b}{x}$, where a and b are positive constants.

b State, giving a reason, the number of real solutions to the equation $x^2(3x - a) - \frac{b}{x} = 0$.

a



b From the sketch, there are only two points of intersection of the curves. This means there are only two values of x where

$$x^2(3x - a) = \frac{b}{x}$$

or $x^2(3x - a) - \frac{b}{x} = 0$

So this equation has two real solutions.

$3x - a = 0$ when $x = \frac{1}{3}a$, so the graph of $y = x^2(3x - a)$ touches the x-axis at (0, 0) and intersects it at $\left(\frac{1}{3}a, 0\right)$.

Problem-solving

You can sketch curves involving unknown constants. You should give any points of intersection with the coordinate axes in terms of the constants where **appropriate**.

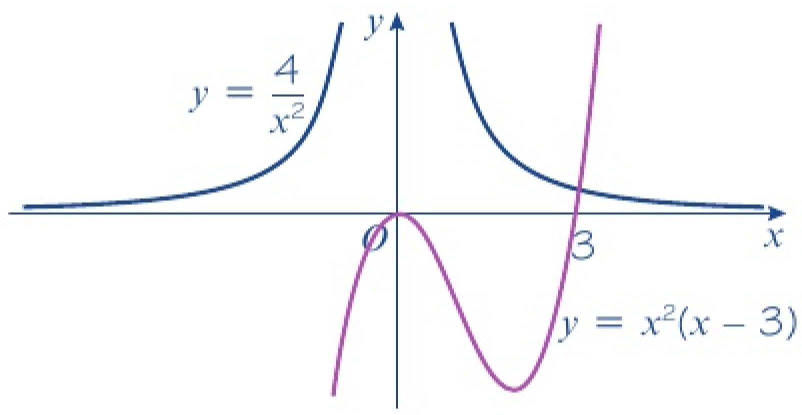
Hint In this question, you only need to state the **number** of solutions. You don't need to find the solutions.

Example 7

a Sketch the curves $y = \frac{4}{x^2}$ and $y = x^2(x - 3)$ on the same axes.

b Using your sketch, state, with a reason, the number of real solutions to the equation.

a



b There is a single point of intersection so the equation $x^2(x - 3) = \frac{4}{x^2}$ has one real solution.

Rearranging:

$x^4(x - 3) = 4$

$x^4(x - 3) - 4 = 0$

So this equation has one real solution.

Problem-solving

Set the functions equal to each other to form an equation with one real solution, then rearrange the equation into the form given in the question.

You would not be expected to solve this equation in your exam.

Exercise 4C

SKILLS INTERPRETATION

1 In each case:

i sketch the two curves on the same axes

ii state the number of points of intersection

iii write down a suitable equation which would give the x-coordinates of these points. (You are not required to solve this equation.)

a $y = x^2, y = x(x^2 - 1)$

b $y = x(x + 2), y = -\frac{3}{x}$

c $y = x^2, y = (x + 1)(x - 1)^2$

d $y = x^2(1 - x), y = -\frac{2}{x}$

e $y = x(x - 4), y = \frac{1}{x}$

f $y = x(x - 4), y = -\frac{1}{x}$

g $y = x(x - 4), y = (x - 2)^3$

h $y = -x^3, y = -\frac{2}{x}$

i $y = -x^3, y = x^2$

j $y = -x^3, y = -x(x + 2)$

2 **a** On the same axes, sketch the curves given by $y = x^2(x - 3)$ and $y = \frac{2}{x}$.

b Explain how your sketch shows that there are only two real solutions to the equation $x^3(x - 3) = 2$.

3 **a** On the same axes, sketch the curves given by $y = (x + 1)^3$ and $y = 3x(x - 1)$.

b Explain how your sketch shows that there is only one real solution to the equation $x^3 + 6x + 1 = 0$.

4 **a** On the same axes, sketch the curves given by $y = \frac{1}{x}$ and $y = -x(x - 1)^2$.

b Explain how your sketch shows that there are no real solutions to the equation $1 + x^2(x - 1)^2 = 0$.

E/P **5** **a** On the same axes, sketch the curves given by $y = x^2(x - a)$ and $y = \frac{b}{x}$, where a and b are both positive constants. **(5 marks)**

b Using your sketch, state, giving a reason, the number of real solutions to the equation $x^4 - ax^3 - b = 0$. **(1 mark)**

Problem-solving

Even though you don't know the values of a and b, you know they are positive, so you know the shapes of the graphs. You can label the point a on the x-axis on your sketch of $y = x^2(x - a)$.

E **6** **a** On the same set of axes, sketch the graphs of $y = \frac{4}{x^2}$ and $y = 3x + 7$. **(3 marks)**

b Write down the number of real solutions to the equation $\frac{4}{x^2} = 3x + 7$. **(1 mark)**

c Show that you can rearrange the equation to give $(x + 1)(x + 2)(3x - 2) = 0$. **(2 marks)**

d Hence determine the exact coordinates of the points of intersection. **(3 marks)**

7 **a** On the same axes, sketch the curve $y = x^3 - 3x^2 - 4x$ and the line $y = 6x$.

b Find the coordinates of the points of intersection.

P **8** **a** On the same axes, sketch the curve $y = (x^2 - 1)(x - 2)$ and the line $y = 14x + 2$.

b Find the coordinates of the points of intersection.

P **9** **a** On the same axes, sketch the curves with equations $y = (x - 2)(x + 2)^2$ and $y = -x^2 - 8$.

b Find the coordinates of the points of intersection.

E/P **10** **a** Sketch the graphs of $y = x^2 + 1$ and $2y = x - 1$. **(3 marks)**

b Explain why there are no real solutions to the equation $2x^2 - x + 3 = 0$. **(2 marks)**

c Work out the range of values of a such that the graphs of $y = x^2 + a$ and $2y = x - 1$ have two points of intersection. **(5 marks)**

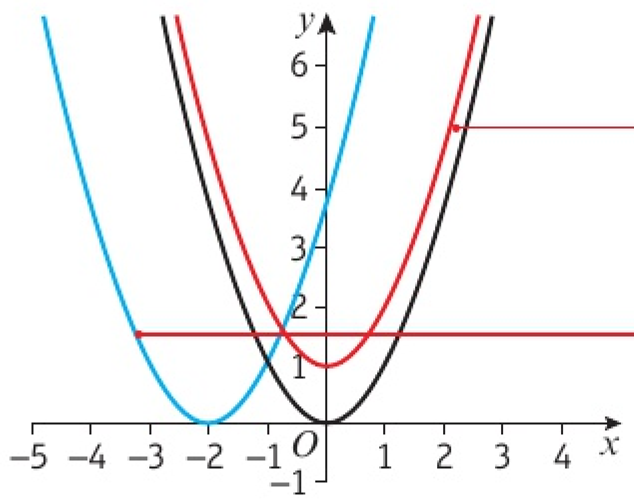
4.4 Translating graphs

You can **transform** the graph of a function by altering the function. Adding or subtracting a constant 'outside' the function **translates** a graph vertically.

- The graph of $y = f(x) + a$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.

Adding or subtracting a constant 'inside' the function translates the graph horizontally.

- The graph of $y = f(x + a)$ is a translation of the graph $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.



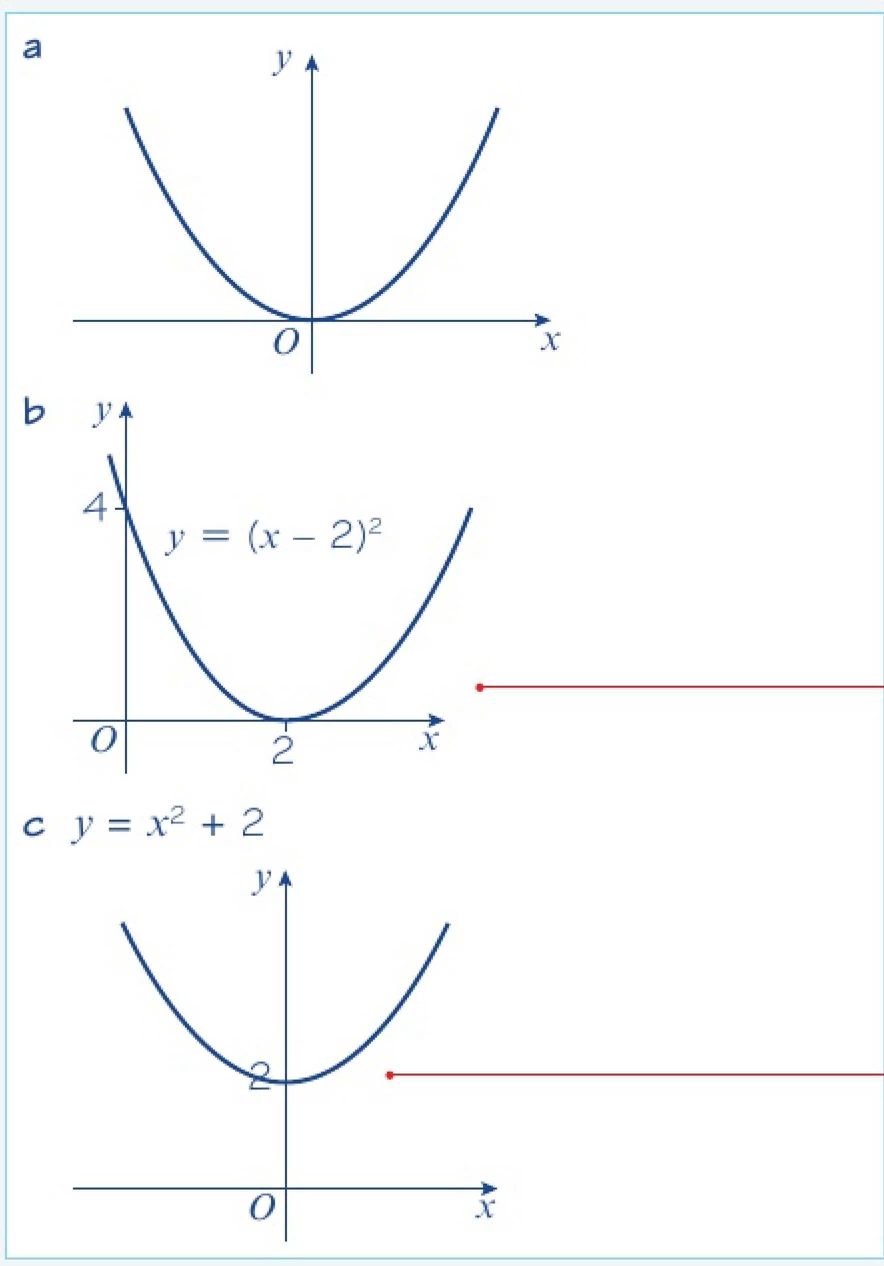
$y = f(x) + 1$ is a translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, or 1 unit in the direction of the positive y-axis.

$y = f(x + 2)$ is a translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or 2 units in the direction of the negative x-axis.

Example 8

Sketch the graphs of:

a $y = x^2$ **b** $y = (x - 2)^2$ **c** $y = x^2 + 2$



This is a translation by vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Remember to mark on the intersections with the axes.

This is a translation by vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Remember to mark on the y-axis intersection.

Example 9

SKILLS INTERPRETATION

$f(x) = x^3$

$g(x) = x(x - 2)$

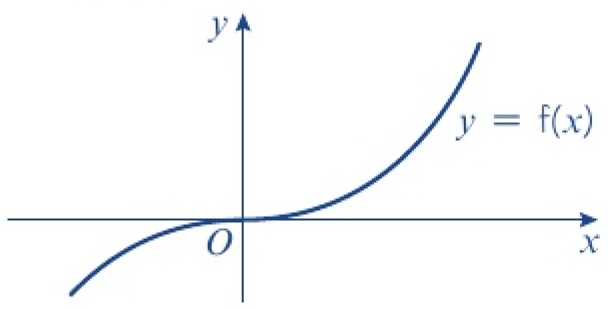
Sketch the following graphs, indicating any points where the curves cross the axes:

a $y = f(x + 1)$

b $y = g(x + 1)$

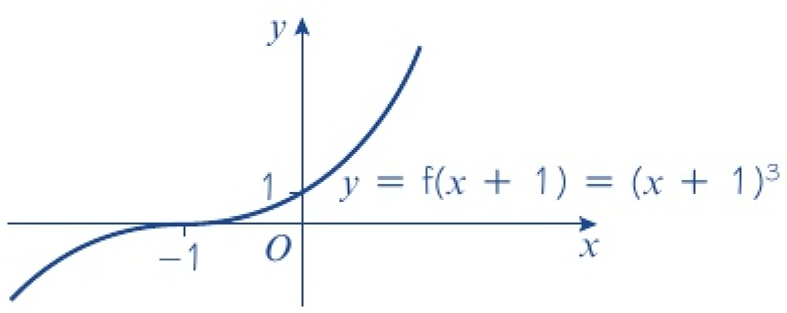
Online Explore translations of the graph of $y = x^3$ using technology.

a The graph of $f(x)$ is



First sketch $y = f(x)$.

So the graph of $y = f(x + 1)$ is



This is a translation of the graph of $y = f(x)$ by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

You could also write out the equation as $y = (x + 1)^3$ and sketch the graph directly.

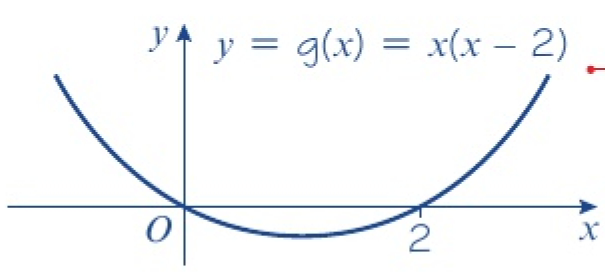
b $g(x) = x(x - 2)$

The curve is $y = x(x - 2)$

$0 = x(x - 2)$

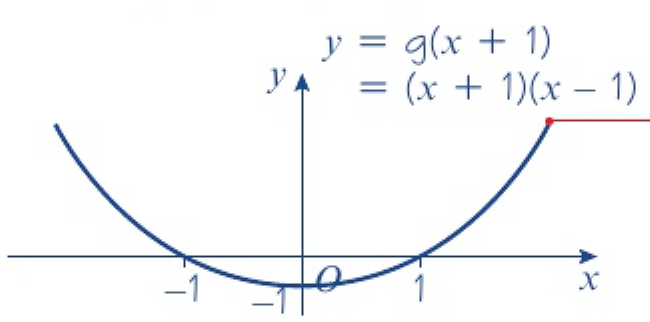
Put $y = 0$ to find where the curve crosses the x-axis.

So $x = 0$ or $x = 2$



First sketch $y = g(x)$.

So the graph of $y = g(x + 1)$ is



This is a translation of the graph of $y = g(x)$ by vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

You could also write out the equation and sketch the graph directly:

$y = g(x + 1)$

$= (x + 1)(x + 1 - 2)$

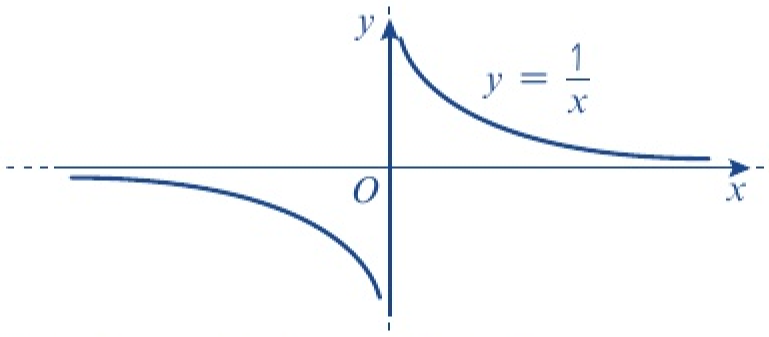
$= (x + 1)(x - 1)$

- When you translate a function, any **asymptotes** are also translated.

Example 10

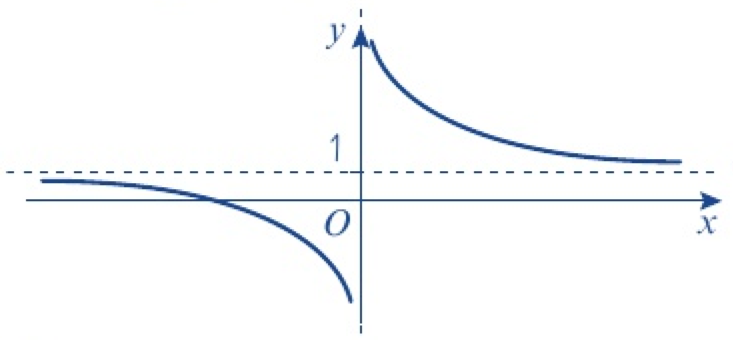
Given that $h(x) = \frac{1}{x}$, sketch the curve with equation $y = h(x) + 1$ and state the equations of any asymptotes and intersections with the axes.

The graph of $y = h(x)$ is



First sketch $y = h(x)$.

So the graph of $y = h(x) + 1$ is



The curve is translated by vector $\begin{pmatrix}0\\1\end{pmatrix}$ so the asymptote is translated by the same vector.

The curve crosses the x-axis once.

$$y = h(x) + 1 = \frac{1}{x} + 1$$

$$0 = \frac{1}{x} + 1$$

$$-1 = \frac{1}{x}$$

$$x = -1$$

Put $y = 0$ to find where the curve crosses the x-axis.

So the curve intersects the x-axis at $(-1, 0)$.

The horizontal asymptote is $y = 1$.

The vertical asymptote is $x = 0$.

Remember to write down the equation of the vertical asymptote as well. It is the y-axis so it has equation $x = 0$.

Exercise 4D SKILLS ANALYSIS

1 Apply the following transformations to the curves with equations $y = f(x)$ where:

i $f(x) = x^2$ **ii** $f(x) = x^3$ **iii** $f(x) = \frac{1}{x}$

In each case state the coordinates of points where the curves cross the axes and in **iii** state the equations of the asymptotes.

a $f(x + 2)$ **b** $f(x) + 2$ **c** $f(x - 1)$

d $f(x) - 1$ **e** $f(x) - 3$ **f** $f(x - 3)$

2 **a** Sketch the curve $y = f(x)$ where $f(x) = (x - 1)(x + 2)$.

b On separate diagrams, sketch the graphs of: **i** $y = f(x + 2)$ **ii** $y = f(x) + 2$.

c Find the equations of the curves $y = f(x + 2)$ and $y = f(x) + 2$, in terms of x, and use these equations to find the coordinates of the points where your graphs in part **b** cross the y-axis.

3 **a** Sketch the graph of $y = f(x)$ where $f(x) = x^2(1 - x)$.

b Sketch the curve with equation $y = f(x + 1)$.

c By finding the equation $f(x + 1)$ in terms of x, find the coordinates of the point in part **b** where the curve crosses the y-axis.

4 **a** Sketch the graph of $y = f(x)$ where $f(x) = x(x - 2)^2$.

b Sketch the curves with equations $y = f(x) + 2$ and $y = f(x + 2)$.

c Find the coordinates of the points where the graph of $y = f(x + 2)$ crosses the axes.

5 **a** Sketch the graph of $y = f(x)$ where $f(x) = x(x - 4)$.

b Sketch the curves with equations $y = f(x + 2)$ and $y = f(x) + 4$.

c Find the equations of the curves in part **b** in terms of x and hence find the coordinates of the points where the curves cross the axes.

E **6** The point $P(4, -1)$ lies on the curve with equation $y = f(x)$.

a State the coordinates that point P is transformed to on the curve with equation $y = f(x - 2)$. **(1 mark)**

b State the coordinates that point P is transformed to on the curve with equation $y = f(x) + 3$. **(1 mark)**

E/P **7** The graph of $y = f(x)$ where $f(x) = \frac{1}{x}$ is translated so that the asymptotes are at $x = 4$ and $y = 0$. Write down the equation for the transformed function in the form $y = \frac{1}{x + a}$. **(3 marks)**

P **8** **a** Sketch the graph of $y = x^3 - 5x^2 + 6x$, marking clearly the points of intersection with the axes.

b Hence sketch $y = (x - 2)^3 - 5(x - 2)^2 + 6(x - 2)$.

E/P **9** **a** Sketch the graph of $y = x^3 + 4x^2 + 4x$. **(6 marks)**

b The point with coordinates $(-1, 0)$ lies on the curve with equation $y = (x + a)^3 + 4(x + a)^2 + 4(x + a)$ where a is a constant. Find the two possible values of a. **(3 marks)**

Problem-solving

Look at your sketch and picture the curve sliding to the left or right.

Challenge

1 The point $Q(-5, -7)$ lies on the curve with equation $y = f(x)$.

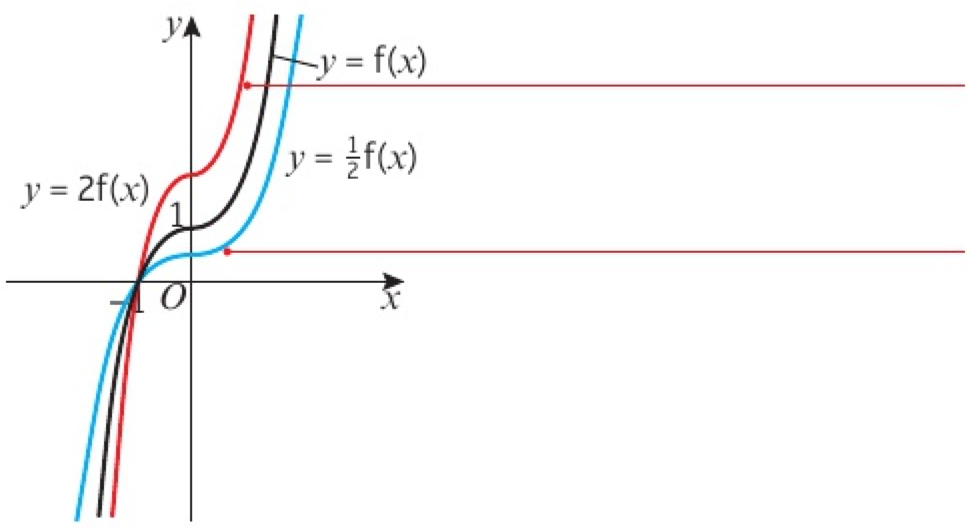
a State the coordinates that point Q is transformed to on the curve with equation $y = f(x + 2) - 5$.

b The coordinates of the point Q on a transformed curve are $(-3, -6)$. Write down the transformation in the form $y = f(x + a) - b$.

4.5 Stretching graphs

Multiplying by a constant 'outside' the function **stretches** the graph vertically.

- The graph of $y = af(x)$ is a stretch of the graph $y = f(x)$ by a **scale** factor of a in the vertical direction.

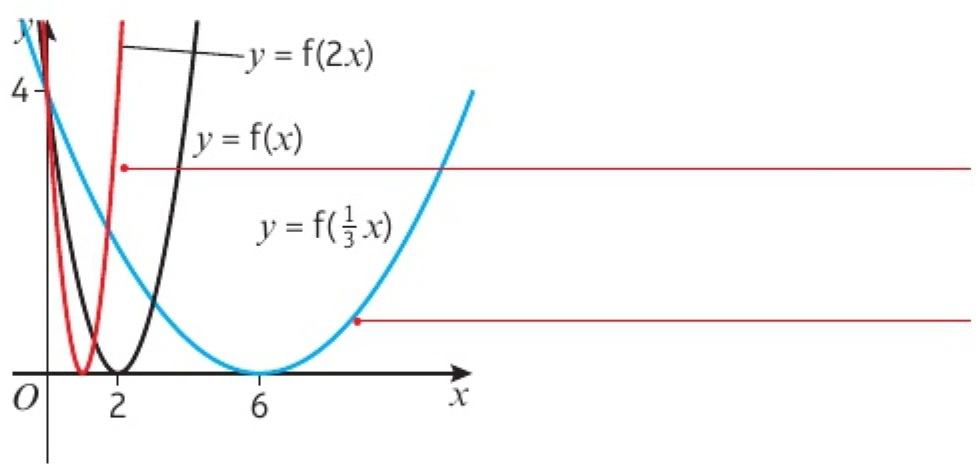


$y = 2f(x)$ is a stretch with scale factor 2 in the y-direction. All y-coordinates are doubled.

$y = \frac{1}{2}f(x)$ is a stretch with scale factor $\frac{1}{2}$ in the y-direction. All y-coordinates are halved.

Multiplying by a constant 'inside' the function stretches the graph horizontally.

- The graph of $y = f(ax)$ is a stretch of the graph $y = f(x)$ by a scale factor of $\dfrac{1}{a}$ in the horizontal direction.



$y = f(2x)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction. All x-coordinates are halved.

$y = f\left(\frac{1}{3}x\right)$ is a stretch with scale factor 3 in the x-direction. All x-coordinates are tripled.

Example 11 SKILLS ANALYSIS

Given that $f(x) = 9 - x^2$, sketch the curves with equations:

a $y = f(2x)$ **b** $y = 2f(x)$

a $f(x) = 9 - x^2$

So $f(x) = (3 - x)(3 + x)$

You can factorise the expression.

The curve is $y = (3 - x)(3 + x)$

$0 = (3 - x)(3 + x)$

Put $y = 0$ to find where the curve crosses the x-axis.

So $x = 3$ or $x = -3$

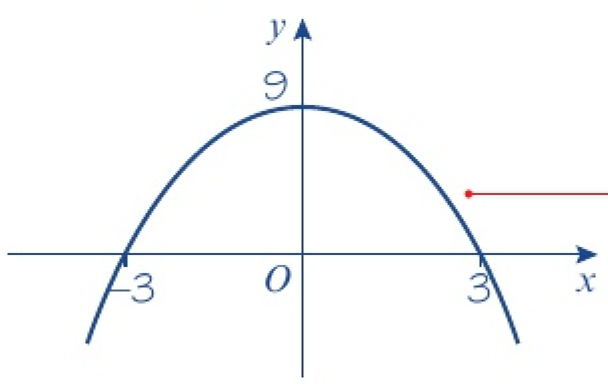
So the curve crosses the x-axis at (3, 0) and (−3, 0).

When $x = 0$, $y = 3 \times 3 = 9$

Put $x = 0$ to find where the curve crosses the y-axis.

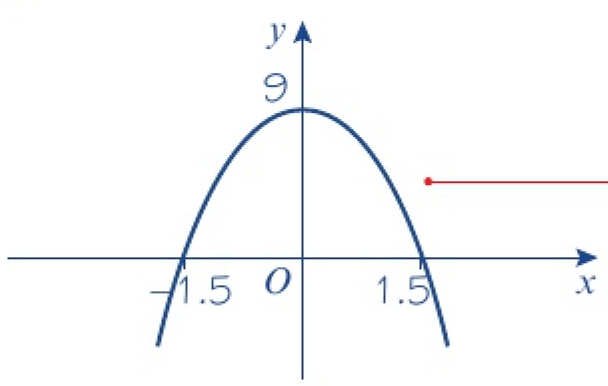
So the curve crosses the y-axis at (0, 9).

The curve $y = f(x)$ is



First sketch $y = f(x)$.

$y = f(2x)$ so the curve is



$y = f(ax)$ where $a = 2$ so it is a horizontal stretch with scale factor $\frac{1}{2}$.

Check: The curve is $y = f(2x)$.

So $y = (3 - 2x)(3 + 2x)$.

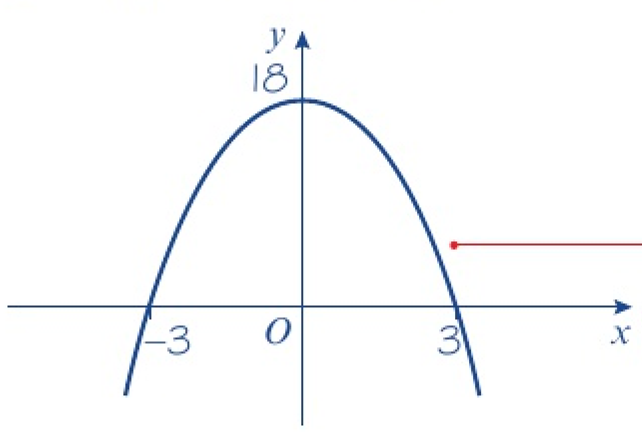
When $y = 0$, $x = -1.5$ or $x = 1.5$.

So the curve crosses the x-axis at (−1.5, 0) and (1.5, 0).

When $x = 0$, $y = 9$.

So the curve crosses the y-axis at (0, 9).

b $y = 2f(x)$ so the curve is



$y = af(x)$ where $a = 2$ so it is a vertical stretch with scale factor 2.

Check: The curve is $y = 2f(x)$.

So $y = 2(3 - x)(3 + x)$.

When $y = 0$, $x = 3$ or $x = -3$.

So the curve crosses the x-axis at (−3, 0) and (3, 0).

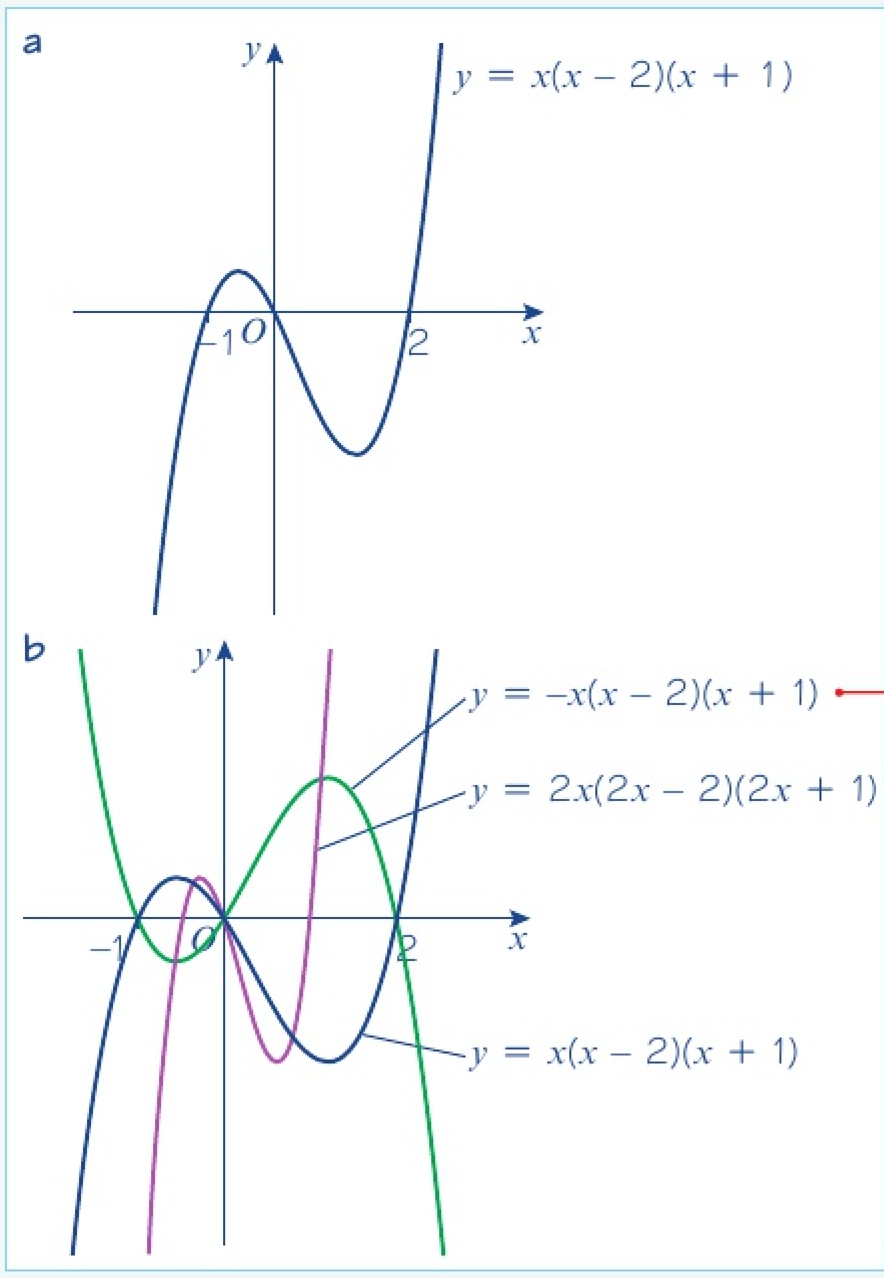
When $x = 0$, $y = 2 \times 9 = 18$.

So the curve crosses the y-axis at (0, 18).

Example 12

a Sketch the curve with equation $y = x(x - 2)(x + 1)$.

b On the same axes, sketch the curves $y = 2x(2x - 2)(2x + 1)$ and $y = -x(x - 2)(x + 1)$.



Online Explore stretches of the graph of $y = x(x - 2)(x + 1)$ using technology.

$y = -x(x - 2)(x + 1)$ is a stretch with scale factor -1 in the y-direction. Notice that this stretch has the effect of reflecting the curve in the x-axis.

$y = 2x(2x - 2)(2x + 1)$ is a stretch with scale factor $\frac{1}{2}$ in the x-direction.

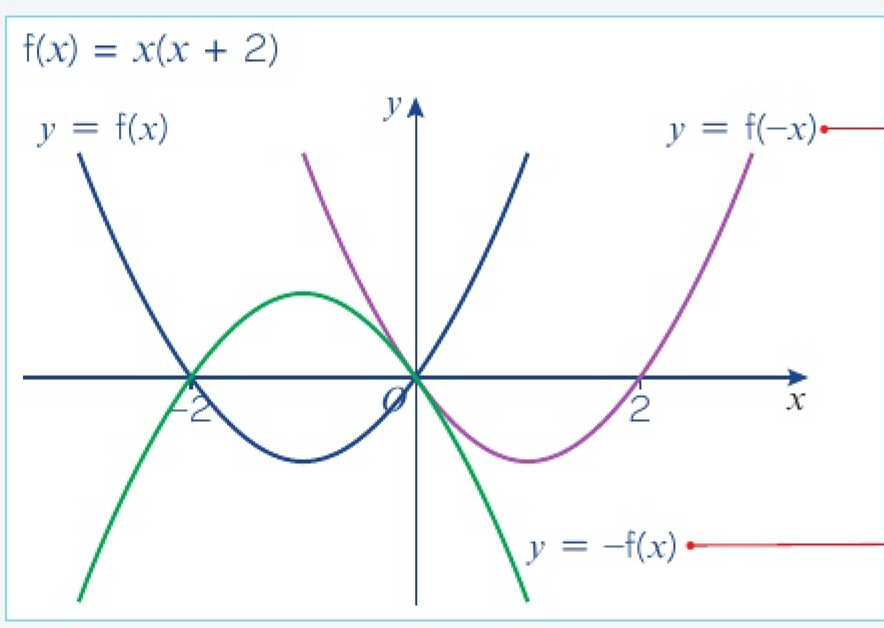
Problem-solving

You need to work out the relationship between each new function and the original function.
If $x(x - 2)(x + 1) = f(x)$ then
$2x(2x - 2)(2x + 1) = f(2x)$, and
$-x(x - 2)(x + 1) = -f(x)$.

- **The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.**
- **The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.**

Example 13

On the same axes, sketch the graphs of $y = f(x)$, $y = f(-x)$ and $y = -f(x)$ where $f(x) = x(x + 2)$.



$y = f(-x)$ is $y = (-x)(-x + 2)$ which is $y = x^2 - 2x$ or $y = x(x - 2)$ and this is a reflection of the original curve in the y-axis.
Alternatively multiply each x-coordinate by -1 and leave the y-coordinates unchanged.
This is the same as a stretch parallel to the x-axis scale factor -1.

$y = -f(x)$ is $y = -x(x + 2)$ and this is a reflection of the original curve in the x-axis.
Alternatively multiply each y-coordinate by -1 and leave the x-coordinates unchanged.
This is the same as a stretch parallel to the y-axis scale factor -1.

Exercise 4E SKILLS ANALYSIS

1 Apply the following transformations to the curves with equations $y = f(x)$ where:

i $f(x) = x^2$ **ii** $f(x) = x^3$ **iii** $f(x) = \frac{1}{x}$

In each case show both f(x) and the transformation on the same diagram.

a $f(2x)$ **b** $f(-x)$ **c** $f(\frac{1}{2}x)$ **d** $f(4x)$ **e** $f(\frac{1}{4}x)$

f $2f(x)$ **g** $-f(x)$ **h** $4f(x)$ **i** $\frac{1}{2}f(x)$ **j** $\frac{1}{4}f(x)$

2 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = x^2 - 4$.

b Sketch the graphs of $y = f(4x)$, $\frac{1}{3}y = f(x)$, $y = f(-x)$ and $y = -f(x)$.

Hint For part **b**, rearrange the second equation into the form $y = 3f(x)$.

3 **a** Sketch the curve with equation $y = f(x)$ where $f(x) = (x - 2)(x + 2)x$.

b Sketch the graphs of $y = f(\frac{1}{2}x)$, $y = f(2x)$ and $y = -f(x)$.

(P) 4 **a** Sketch the curve with equation $y = x^2(x - 3)$.

b On the same axes, sketch the curves with equations:

i $y = (2x)^2(2x - 3)$ **ii** $y = -x^2(x - 3)$

Problem-solving Let $f(x) = x^2(x - 3)$ and try to write each of the equations in part **b** in terms of f(x).

5 **a** Sketch the curve $y = x^2 + 3x - 4$.

b On the same axes, sketch the graph of $5y = x^2 + 3x - 4$.

6 **a** Sketch the graph of $y = x^2(x - 2)^2$.

b On the same axes, sketch the graph of $3y = -x^2(x - 2)^2$.

(E) 7 The point $P(2, -3)$ lies on the curve with equation $y = f(x)$.

a State the coordinates that point P is transformed to on the curve with equation $y = f(2x)$. **(1 mark)**

b State the coordinates that point P is transformed to on the curve with equation $y = 4f(x)$. **(1 mark)**

(E) 8 The point $Q(-2, 8)$ lies on the curve with equation $y = f(x)$.

State the coordinates that point Q is transformed to on the curve with equation $y = f(\frac{1}{2}x)$. **(1 mark)**

(E/P) 9 **a** Sketch the graph of $y = (x - 2)(x - 3)^2$. **(4 marks)**

b The graph of $y = (ax - 2)(ax - 3)^2$ passes through the point (1, 0). Find two possible values for a. **(3 marks)**

Challenge

1 The point $R(4, -6)$ lies on the curve with equation $y = f(x)$. State the coordinates that point R is transformed to on the curve with equation $y = \frac{1}{3}f(2x)$.

2 The point $S(-4, 7)$ is transformed to a point $S'(-8, 1.75)$. Write down the transformation in the form $y = af(bx)$.

4.6 Transforming functions

You can apply transformations to unfamiliar functions by considering how specific points and features are transformed.

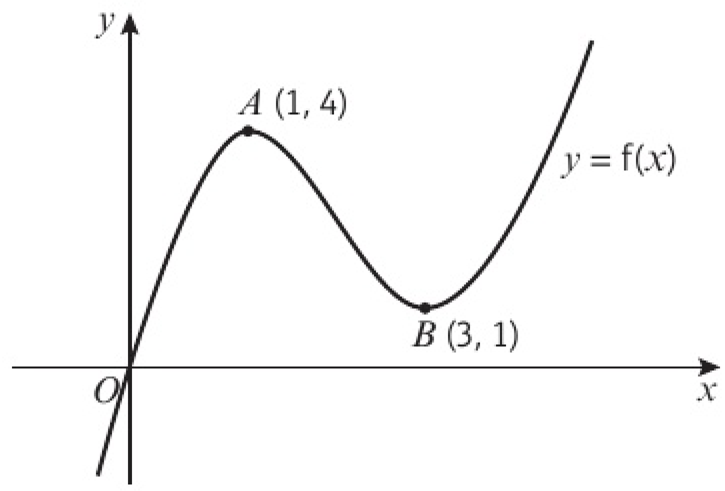
Example 14

The diagram shows a sketch of the curve f(x) which passes through the origin.
The points $A(1, 4)$ and $B(3, 1)$ also **lie on** the curve.

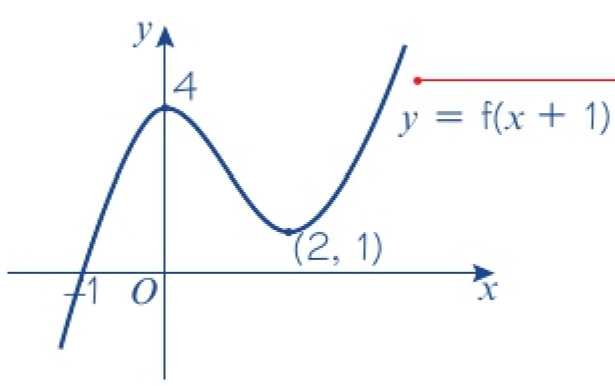
Sketch the following:

a $y = f(x + 1)$ **b** $y = f(x - 1)$ **c** $y = f(x) - 4$
d $2y = f(x)$ **e** $y - 1 = f(x)$

In each case you should show the positions of the images of the points O, A and B.

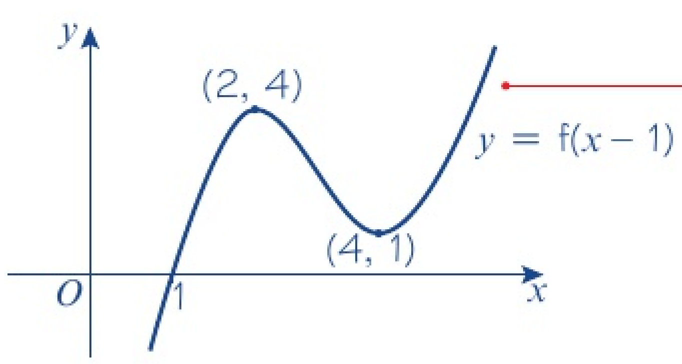


a f(x + 1)



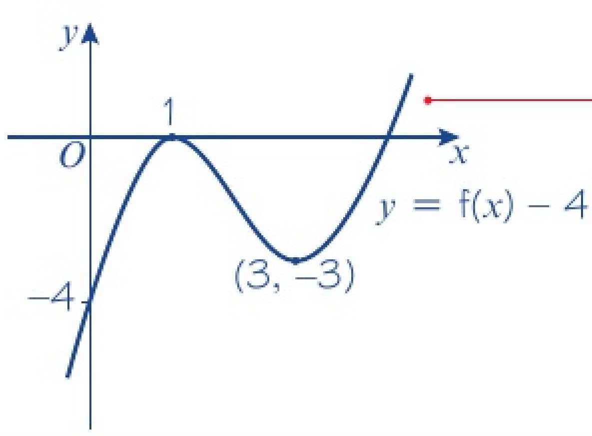
Translate f(x) 1 unit in the direction of the negative x-axis.

b f(x − 1)



Translate f(x) 1 unit in the direction of the positive x-axis.

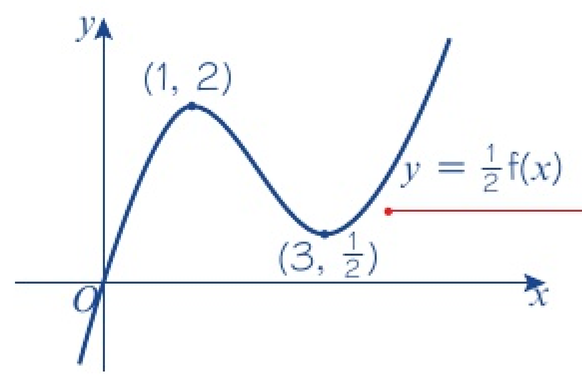
c f(x) − 4



Translate f(x) 4 units in the direction of the negative y-axis.

d $2y = f(x)$ so $y = \frac{1}{2}f(x)$

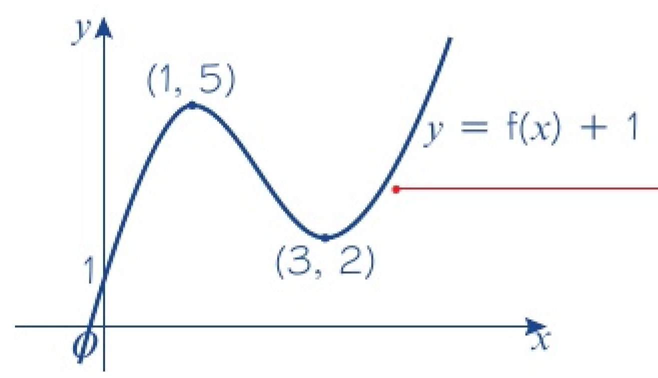
Rearrange in the form $y = \ldots$



Stretch $f(x)$ by scale factor $\frac{1}{2}$ in the y-direction.

e $y - 1 = f(x)$ so $y = f(x) + 1$

Rearrange in the form $y = \ldots$

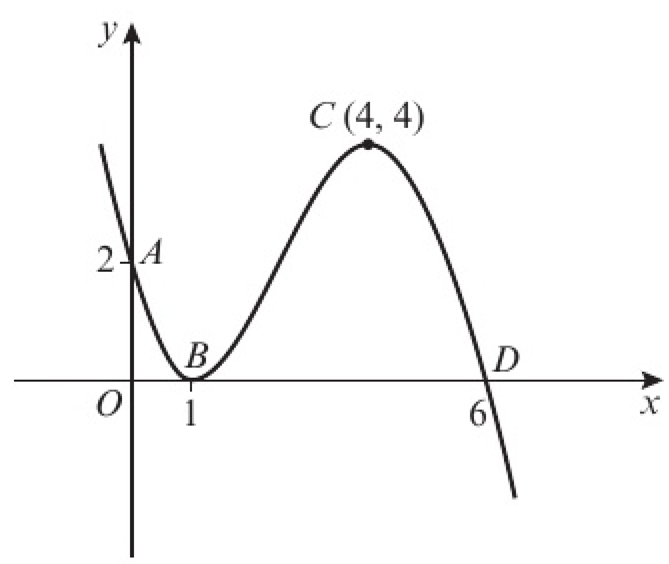


Translate $f(x)$ 1 unit in the direction of the positive y-axis.

Exercise 4F SKILLS ANALYSIS

1 The following diagram shows a sketch of the curve with equation $y = f(x)$. The points $A(0, 2)$, $B(1, 0)$, $C(4, 4)$ and $D(6, 0)$ lie on the curve.

Sketch the following graphs and give the coordinates of the points A, B, C and D after each transformation:



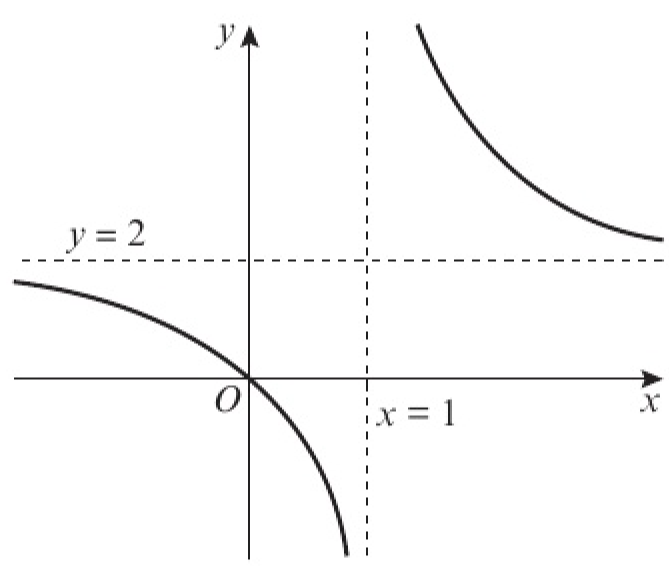
a $f(x + 1)$ **b** $f(x) - 4$ **c** $f(x + 4)$

d $f(2x)$ **e** $3f(x)$ **f** $f(\frac{1}{2}x)$

g $\frac{1}{2}f(x)$ **h** $f(-x)$

2 The curve $y = f(x)$ passes through the origin and has horizontal asymptote $y = 2$ and vertical asymptote $x = 1$, as shown in the diagram.

Sketch the following graphs. Give the equations of any asymptotes and give the coordinates of intersections with the axes after each transformation.



a $f(x) + 2$ **b** $f(x + 1)$ **c** $2f(x)$

d $f(x) - 2$ **e** $f(2x)$ **f** $f(\frac{1}{2}x)$

g $\frac{1}{2}f(x)$ **h** $-f(x)$

3 The curve with equation $y = f(x)$ passes through the points $A(-4, -6)$, $B(-2, 0)$, $C(0, -3)$ and $D(4, 0)$ as shown in the diagram.

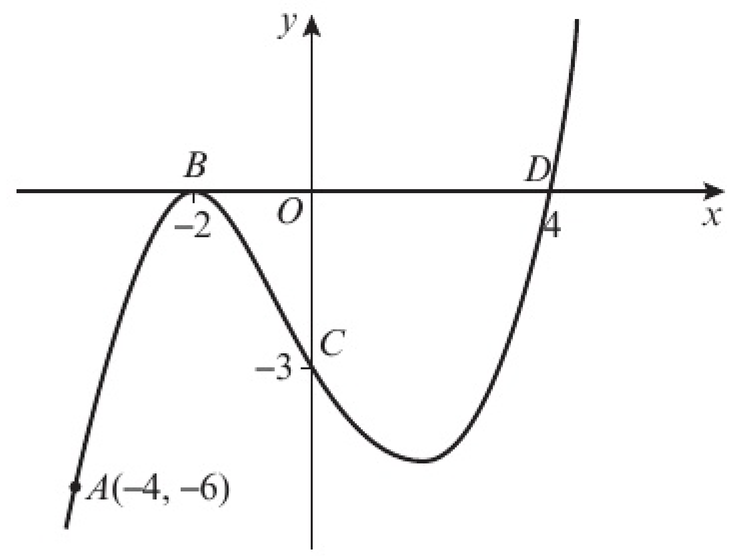
Sketch the following and give the coordinates of the points A, B, C and D after each transformation.

a $f(x - 2)$ **b** $f(x) + 6$ **c** $f(2x)$

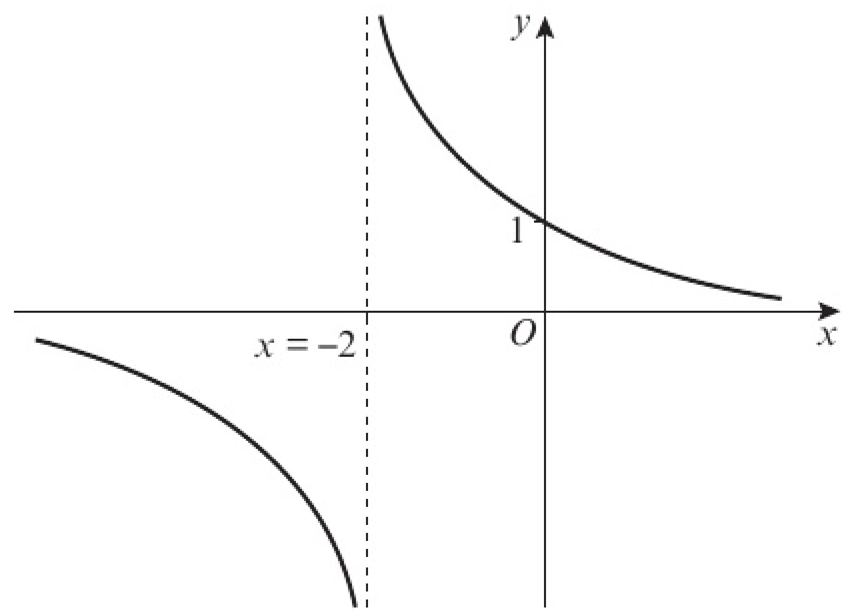
d $f(x + 4)$ **e** $f(x) + 3$ **f** $3f(x)$

g $\frac{1}{3}f(x)$ **h** $f(\frac{1}{4}x)$ **i** $-f(x)$

j $f(-x)$



4 A sketch of the curve $y = f(x)$ is shown in the diagram. The curve has a vertical asymptote with equation $x = -2$ and a horizontal asymptote with equation $y = 0$. The curve crosses the y-axis at $(0, 1)$.



a Sketch, on separate diagrams, the graphs of:

i $2f(x)$ **ii** $f(2x)$ **iii** $f(x - 2)$

iv $f(x) - 1$ **v** $f(-x)$ **vi** $-f(x)$

In each case state the equations of any asymptotes and, if possible, points where the curve cuts the axes.

b Suggest a possible equation for $f(x)$.

E/P **5** The point $P(2, 1)$ lies on the graph with equation $y = f(x)$.

a On the graph of $y = f(ax)$, the point P is transformed to the point $Q(4, 1)$. Determine the value of a. **(1 mark)**

b Write down the coordinates of the point to which P maps under each transformation.

i $f(x - 4)$ **ii** $3f(x)$ **iii** $\frac{1}{2}f(x) - 4$ **(3 marks)**

P **6** The diagram shows a sketch of a curve with equation $y = f(x)$. The points $A(-1, 0)$, $B(0, 2)$, $C(1, 2)$ and $D(2, 0)$ lie on the curve.

Sketch the following graphs and give the coordinates of the points A, B, C and D after each transformation:

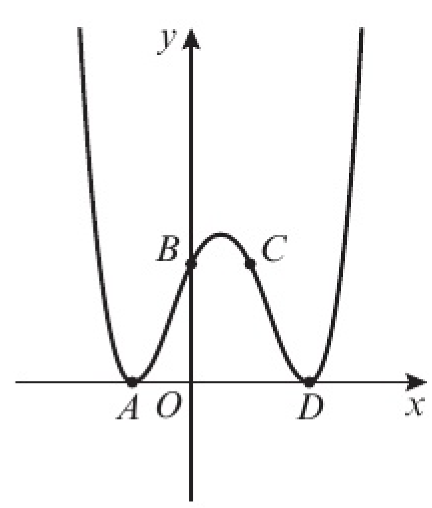
a $y + 2 = f(x)$ **b** $\frac{1}{2}y = f(x)$

c $y - 3 = f(x)$ **d** $3y = f(x)$

e $2y - 1 = f(x)$

Problem-solving

Rearrange each equation into the form $y = \ldots$



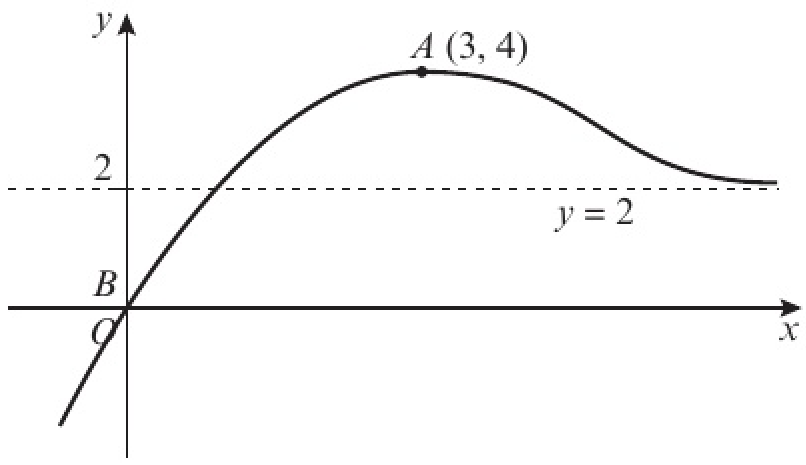
Chapter review 4

SKILLS EXECUTIVE FUNCTION

1 a On the same axes, sketch the graphs of $y = x^2(x - 2)$ and $y = 2x - x^2$.
b By solving a suitable equation, find the points of intersection of the two graphs.

P 2 a On the same axes, sketch the curves with equations $y = \frac{6}{x}$ and $y = 1 + x$.
b The curves intersect at the points A and B. Find the coordinates of A and B.
c The curve C with equation $y = x^2 + px + q$, where p and q are integers, passes through A and B. Find the values of p and q.
d Add C to your sketch.

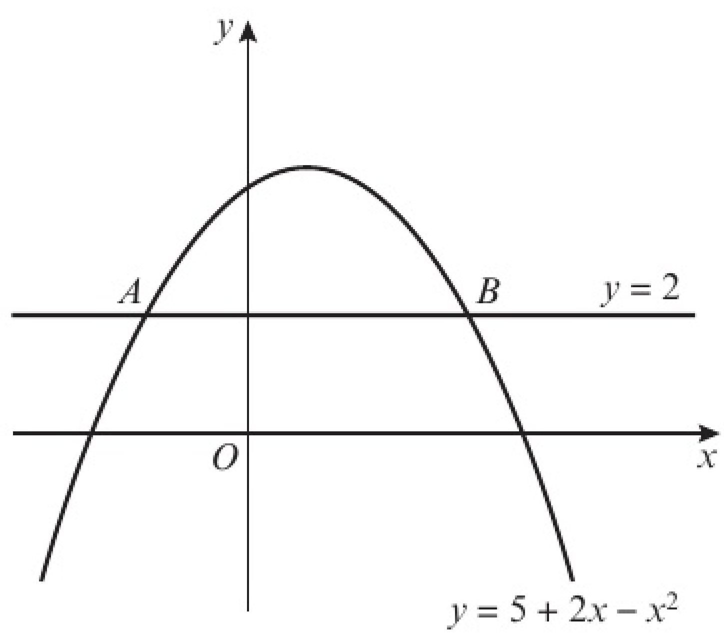
3 The diagram shows a sketch of the curve $y = f(x)$.
The point $B(0, 0)$ lies on the curve and the point $A(3, 4)$ is a maximum point.
The line $y = 2$ is an asymptote.



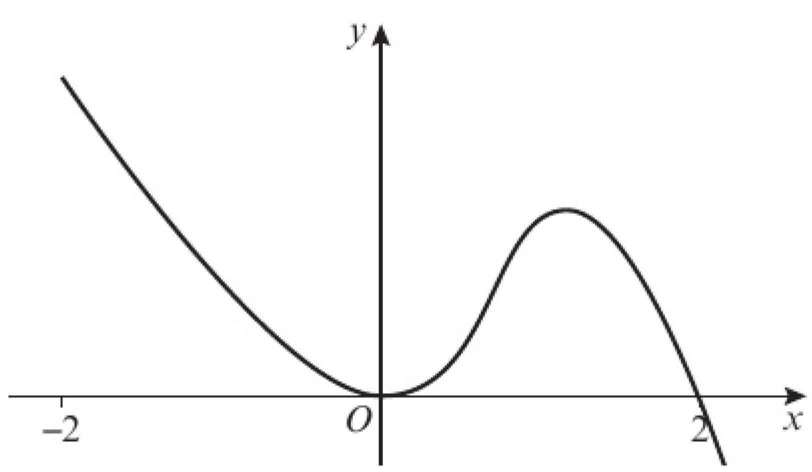
Sketch the following, and in each case give the coordinates of the new positions of A and B and state the equation of the asymptote:

a $f(2x)$ b $\frac{1}{2}f(x)$ c $f(x) - 2$
d $f(x + 3)$ e $f(x - 3)$ f $f(x) + 1$

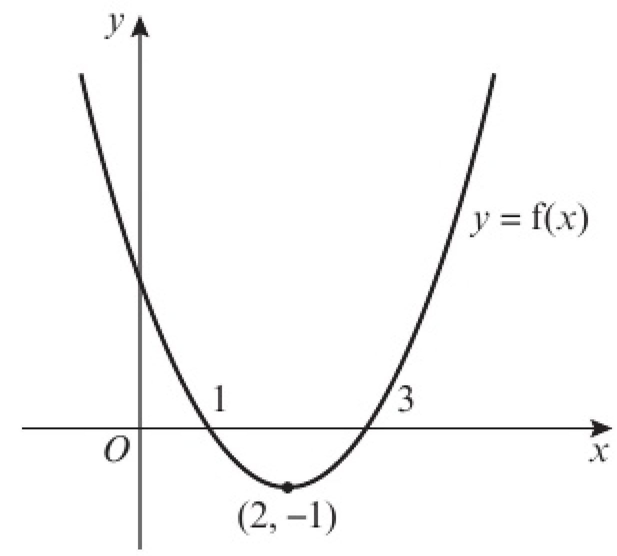
E 4 The diagram shows the curve with equation $y = 5 + 2x - x^2$ and the line with equation $y = 2$. The curve and the line intersect at the points A and B.
Find the x-coordinates of A and B. **(4 marks)**



E 5 The diagram shows a sketch of the curve with equation $y = f(x)$.
On separate axes, sketch the curves with equations:
a $y = f(-x)$ **(2 marks)**
b $y = -f(x)$ **(2 marks)**
Mark on each sketch the x-coordinate of any point, or points, where the curve touches or crosses the x-axis.



E/P **6** The diagram shows the graph of the quadratic function f(x). The graph meets the x-axis at (1, 0) and (3, 0) and the minimum point is (2, −1).



a Find the equation of the graph in the form $y = ax^2 + bx + c$. **(2 marks)**

b On separate axes, sketch the graphs of
i $y = \mathrm{f}(x + 2)$ **ii** $y = \mathrm{f}(2x)$. **(2 marks)**

c On each graph, label the coordinates of the points at which the graph meets the x-axis and label the coordinates of the minimum point.

E/P **7** $\mathrm{f}(x) = (x - 1)(x - 2)(x + 1)$.

a State the coordinates of the point at which the graph $y = \mathrm{f}(x)$ intersects the y-axis. **(1 mark)**

b The graph of $y = a\mathrm{f}(x)$ intersects the y-axis at (0, −4). Find the value of a. **(1 mark)**

c The graph of $y = \mathrm{f}(x + b)$ passes through the origin. Find three possible values of b. **(3 marks)**

P **8** The point $P(4, 3)$ lies on a curve with equation $y = \mathrm{f}(x)$.

a State the coordinates of the point to which P is transformed on the curve with equation:
i $y = \mathrm{f}(3x)$ **ii** $\frac{1}{2}y = \mathrm{f}(x)$ **iii** $y = \mathrm{f}(x - 5)$ **iv** $-y = \mathrm{f}(x)$ **v** $2(y + 2) = \mathrm{f}(x)$

b P is transformed to point (2, 3). Write down two possible transformations of f(x).

c P is transformed to point (8, 6). Write down a possible transformation of f(x) if
i f(x) is translated only **ii** f(x) is stretched only.

E/P **9** **a** Factorise completely $x^3 - 6x^2 + 9x$. **(2 marks)**

b Sketch the curve of $y = x^3 - 6x^2 + 9x$, showing clearly the coordinates of the points where the curve touches or crosses the axes. **(4 marks)**

c The point with coordinates (−4, 0) lies on the curve with equation $y = (x - k)^3 - 6(x - k)^2 + 9(x - k)$ where k is a constant.
Find the two possible values of k. **(3 marks)**

E **10** $\mathrm{f}(x) = x(x - 2)^2$

Sketch, on separate axes, the graphs of:

a $y = \mathrm{f}(x)$ **(2 marks)**

b $y = \mathrm{f}(x + 3)$ **(2 marks)**

Show on each sketch the coordinates of the points where each graph crosses or meets the axes.

E **11** Given that $\mathrm{f}(x) = \frac{1}{x}, x \neq 0$

a Sketch the graph of $y = \mathrm{f}(x) - 2$ and state the equations of the asymptotes. **(3 marks)**

b Find the coordinates of the point where the curve $y = \mathrm{f}(x) - 2$ cuts a coordinate axis. **(2 marks)**

c Sketch the graph of $y = \mathrm{f}(x + 3)$. **(2 marks)**

d State the equations of the asymptotes and the coordinates of the point where the curve cuts a coordinate axis. **(2 marks)**

Challenge

The point $R(6, -4)$ lies on the curve with equation $y = \mathrm{f}(x)$. State the coordinates that point R is transformed to on the curve with equation $y = \mathrm{f}(x + c) - d$.

Summary of key points

1. If p is a root of the function $\mathrm{f}(x)$, then the graph of $y = \mathrm{f}(x)$ touches or crosses the x-axis at the point $(p, 0)$.
2. The graphs of $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$, where k is a real constant, have asymptotes at $x = 0$ and $y = 0$.
3. The x-coordinate(s) at the point(s) of intersection of the curves with equations $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ are the solution(s) to the equation $\mathrm{f}(x) = \mathrm{g}(x)$.
4. The graph of $y = \mathrm{f}(x) + a$ is a translation of the graph $y = \mathrm{f}(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$.
5. The graph of $y = \mathrm{f}(x + a)$ is a translation of the graph $y = \mathrm{f}(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.
6. When you translate a function, any asymptotes are also translated.
7. The graph of $y = a\mathrm{f}(x)$ is a stretch of the graph $y = \mathrm{f}(x)$ by a scale factor of a in the vertical direction.
8. The graph of $y = \mathrm{f}(ax)$ is a stretch of the graph $y = \mathrm{f}(x)$ by a scale factor of $\frac{1}{a}$ in the horizontal direction.
9. The graph of $y = -\mathrm{f}(x)$ is a reflection of the graph of $y = \mathrm{f}(x)$ in the x-axis.
10. The graph of $y = \mathrm{f}(-x)$ is a reflection of the graph of $y = \mathrm{f}(x)$ in the y-axis.

Review exercise 1

E **1 Do not use your calculator for this question.**

a Write down the value of $8^{\frac{1}{3}}$. **(1)**

b Find the value of $8^{-\frac{2}{3}}$. **(2)**

← Section 1.4

2 Do not use your calculator for this question.

a Find the value of $125^{\frac{4}{3}}$. **(2)**

b Simplify $24x^2 \div 18x^{\frac{4}{3}}$. **(2)**

← Sections 1.1, 1.4

E **3 Do not use your calculator for this question.**

a Express $\sqrt{80}$ in the form $a\sqrt{5}$, where a is an integer. **(2)**

b Express $(4 - \sqrt{5})^2$ in the form $b + c\sqrt{5}$, where b and c are integers. **(2)**

← Section 1.5

E **4 Do not use your calculator for this question.**

a Expand and simplify $(4 + \sqrt{3})(4 - \sqrt{3})$. **(2)**

b Express $\dfrac{26}{4 + \sqrt{3}}$ in the form $a + b\sqrt{3}$, where a and b are integers. **(3)**

← Sections 1.5, 1.6

E/P **5** Here are three numbers:

$$1 - \sqrt{k}\,,\ 2 + 5\sqrt{k} \text{ and } 2\sqrt{k}$$

Given that k is a positive integer, find:

a the mean of the three numbers **(2)**

b the range of the three numbers. **(1)**

← Section 1.5

E **6** Given that $y = \frac{1}{25}x^4$, express each of the following in the form kx^n, where k and n are constants.

a y^{-1} **(1)**

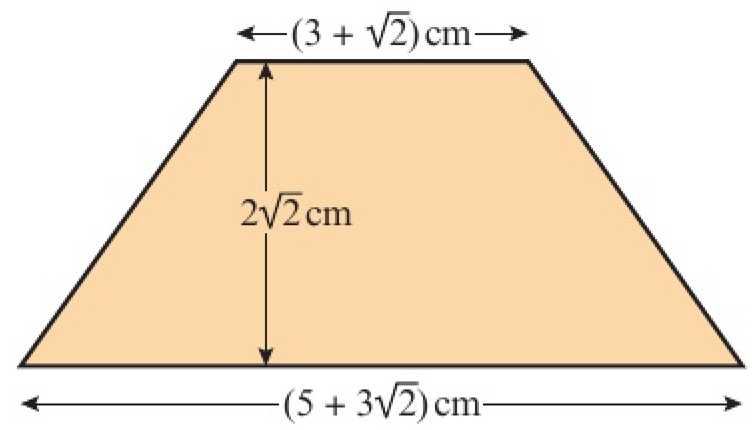
b $5y^{\frac{1}{2}}$ **(1)**

← Section 1.4

E/P **7 Do not use your calculator for this question.**

Find the area of this trapezium in cm^2. Give your answer in the form $a + b\sqrt{2}$, where a and b are integers to be found. **(4)**

← Section 1.5



E **8 Do not use your calculator for this question.**

Given that $p = 3 - 2\sqrt{2}$ and $q = 2 - \sqrt{2}$, find the value of $\dfrac{p + q}{p - q}$.

Give your answer in the form $m + n\sqrt{2}$, where m and n are rational numbers to be found. **(4)**

← Sections 1.5, 1.6

E/P **9** **a** Factorise the expression $x^2 - 10x + 16$. **(1)**

b Hence, or otherwise, solve the equation $8^{2y} - 10(8^y) + 16 = 0$. **(2)**

← Sections 1.3, 2.1

E **10** $x^2 - 8x - 29 \equiv (x + a)^2 + b$, where a and b are constants.

a Find the value of a and the value of b. **(2)**

b Hence, or otherwise, show that the roots of $x^2 - 8x - 29 = 0$ are $c \pm d\sqrt{5}$, where c and d are integers. **(3)**

← Sections 2.1, 2.2

E/P **11** The functions f and g are defined as $f(x) = x(x - 2)$ and $g(x) = x + 5, x \in \mathbb{R}$.
Given that $f(a) = g(a)$ and $a > 0$, find the value of a to three significant figures. **(3)**

← Sections 2.1, 2.3

E/P **12** **a** Given that $f(x) = x^2 - 6x + 18, x \geqslant 0$, express $f(x)$ in the form $(x - a)^2 + b$, where a and b are integers. **(2)**

The curve C with equation $y = f(x)$, $x \geqslant 0$, meets the y-axis at P and has a minimum point at Q.

b Sketch the graph of C, showing the coordinates of P and Q. **(3)**

The line $y = 41$ meets C at the point R.

c Find the x-coordinate of R, giving your answer in the form $p + q\sqrt{2}$, where p and q are integers. **(2)**

← Sections 2.2, 2.4

E **13** The function $h(x) = x^2 + 2\sqrt{2}x + k$ has equal roots.

a Find the value of k. **(1)**

b Sketch the graph of $y = h(x)$, clearly labelling any intersections with the coordinate axes. **(3)**

← Sections 1.5, 2.4, 2.5

E/P **14** The function $g(x)$ is defined as

$$g(x) = x^9 - 7x^6 - 8x^3, x \in \mathbb{R}.$$

a Write $g(x)$ in the form $x^3(x^3 + a)(x^3 + b)$, where a and b are integers. **(1)**

b Hence find the three roots of $g(x)$. **(1)**

← Section 2.3

E/P **15** Given that

$$x^2 + 10x + 36 \equiv (x + a)^2 + b,$$

where a and b are constants,

a find the value of a and the value of b. **(2)**

b Hence show that the equation $x^2 + 10x + 36 = 0$ has no real roots. **(2)**

The equation $x^2 + 10x + k = 0$ has equal roots.

c Find the value of k. **(2)**

d For this value of k, sketch the graph of $y = x^2 + 10x + k$, showing the coordinates of any points at which the graph meets the coordinate axes. **(3)**

← Sections 2.2, 2.4, 2.5

E/P **16** Given that $x^2 + 2x + 3 \equiv (x + a)^2 + b$,

a find the value of the constants a and b. **(2)**

b Sketch the graph of $y = x^2 + 2x + 3$, indicating clearly the coordinates of any intersections with the coordinate axes. **(3)**

c Find the value of the discriminant of $x^2 + 2x + 3$. Explain how the sign of the discriminant relates to your sketch in part **b**. **(2)**

The equation $x^2 + kx + 3 = 0$, where k is a constant, has no real roots.

d Find the set of possible values of k, giving your answer in surd form. **(2)**

← Section 2.2, 2.4, 2.5

E **17** **a** By eliminating y from the equations:

$$y = x - 4$$
$$2x^2 - xy = 8$$

show that

$$x^2 + 4x - 8 = 0.$$ **(2)**

b Hence, or otherwise, solve the simultaneous equations:

$$y = x - 4$$
$$2x^2 - xy = 8$$

giving your answers in the form $a \pm b\sqrt{3}$, where a and b are integers. **(4)**

← Section 3.2

E **18** Find the set of values of x for which:

a $3(2x + 1) > 5 - 2x$ **(2)**

b $2x^2 - 7x + 3 > 0$ **(3)**

c **both** $3(2x + 1) > 5 - 2x$ **and** $2x^2 - 7x + 3 > 0$. **(1)**

← Sections 3.4, 3.5

E/P **19** The functions p and q are defined as $p(x) = -2(x + 1)$ and $q(x) = x^2 - 5x + 2$, $x \in \mathbb{R}$. Show algebraically that there is no value of x for which $p(x) = q(x)$. **(3)**

← Sections 2.3, 2.5

E **20** **a** Solve the simultaneous equations:

$$y + 2x = 5$$
$$2x^2 - 3x - y = 16$$ **(5)**

b Hence, or otherwise, find the set of values of x for which:

$$2x^2 - 3x - 16 > 5 - 2x$$ **(2)**

← Sections 3.2, 3.5

E/P **21** The equation $x^2 + kx + (k + 3) = 0$, where k is a constant, has different real roots.

a Show that $k^2 - 4k - 12 > 0$. **(2)**

b Find the set of possible values of k. **(2)**

← Sections 2.5, 3.5

E **22** Find the set of values for which

$$\frac{6}{x + 5} < 2, x \neq -5.$$ **(6)**

← Section 3.4

E **23** The functions f and g are defined as $f(x) = 9 - x^2$ and $g(x) = 14 - 6x$, $x \in \mathbb{R}$.

a On the same set of axes, sketch the graphs of $y = f(x)$ and $y = g(x)$. Indicate clearly the coordinates of any points where the graphs intersect with each other or the coordinate axes. **(5)**

b On your sketch, shade the region that satisfies the inequalities $y > 0$ and $f(x) > g(x)$. **(1)**

← Sections 3.2, 3.3, 3.7

E/P **24** **a** Factorise completely $x^3 - 4x$. **(1)**

b Sketch the curve with equation $y = x^3 - 4x$, showing the coordinates of the points where the curve crosses the x-axis. **(2)**

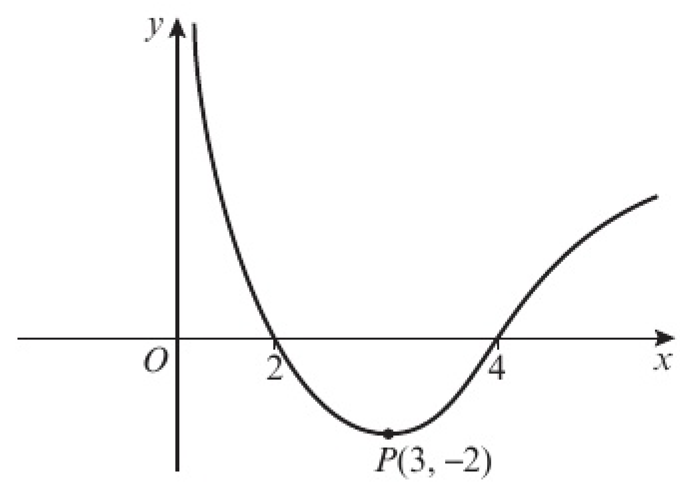
c On a separate diagram, sketch the curve with equation

$$y = (x - 1)^3 - 4(x - 1)$$

showing the coordinates of the points where the curve crosses the x-axis. **(2)**

← Sections 1.3, 4.1, 4.5

E **25**



The figure shows a sketch of the curve with equation $y = f(x)$. The curve crosses the x-axis at the points (2, 0) and (4, 0). The minimum point on the curve is $P(3, -2)$.

In separate diagrams, sketch the curves with equation

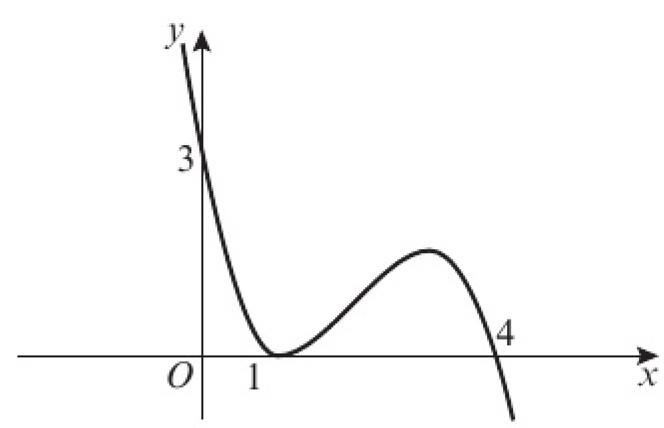
a $y = -f(x)$ **(2)**

b $y = f(2x)$ **(2)**

On each diagram, give the coordinates of the points at which the curve crosses the x-axis, and the coordinates of the image of P under the given transformation.

← Sections 4.5, 4.6

E **26**



The figure shows a sketch of the curve with equation $y = f(x)$. The curve passes through the points (0, 3) and (4, 0) and touches the x-axis at the point (1, 0).

On separate diagrams, sketch the curves with equations

a $y = f(x + 1)$ **(2)**

b $y = 2f(x)$ **(2)**

c $y = f\left(\frac{1}{2}x\right)$ **(2)**

On each diagram, show clearly the coordinates of all the points where the curve meets the axes.

← Sections 4.4, 4.5, 4.6

E **27** Given that $f(x) = \frac{1}{x}, x \neq 0$,

a sketch the graph of $y = f(x) + 3$ and state the equations of the asymptotes **(2)**

b find the coordinates of the point where $y = f(x) + 3$ crosses a coordinate axis. **(2)**

← Sections 4.2, 4.4

E **28** The point $(6, -8)$ lies on the graph of $y = f(x)$. State the coordinates of the point to which P is transformed on the graph with equation:

a $y = -f(x)$ **(1)**

b $y = f(x - 3)$ **(1)**

c $2y = f(x)$ **(1)**

← Section 4.6

E/P **29** The curve C_1 has equation $y = -\frac{a}{x}$, where a is a positive constant.

The curve C_2 has equation $y = (x - b)^2$, where b is a positive constant.

a Sketch C_1 and C_2 on the same set of axes. Label any points where either curve meets the coordinate axes, giving your coordinates in terms of a and b. **(4)**

b Using your sketch, state the number of real solutions to the equation $x(x - 5)^2 = -7$. **(1)**

← Sections 4.2, 4.3

E/P **30** **a** Sketch the graph of $y = \frac{1}{x^2} - 4$, showing clearly the coordinates of the points where the curve crosses the coordinate axes and stating the equations of the asymptotes. **(4)**

b The curve with equation $y = \frac{1}{(x + k)^2} - 4$ passes through the origin. Find the two possible values of k. **(2)**

← Sections 4.2, 4.4, 4.6

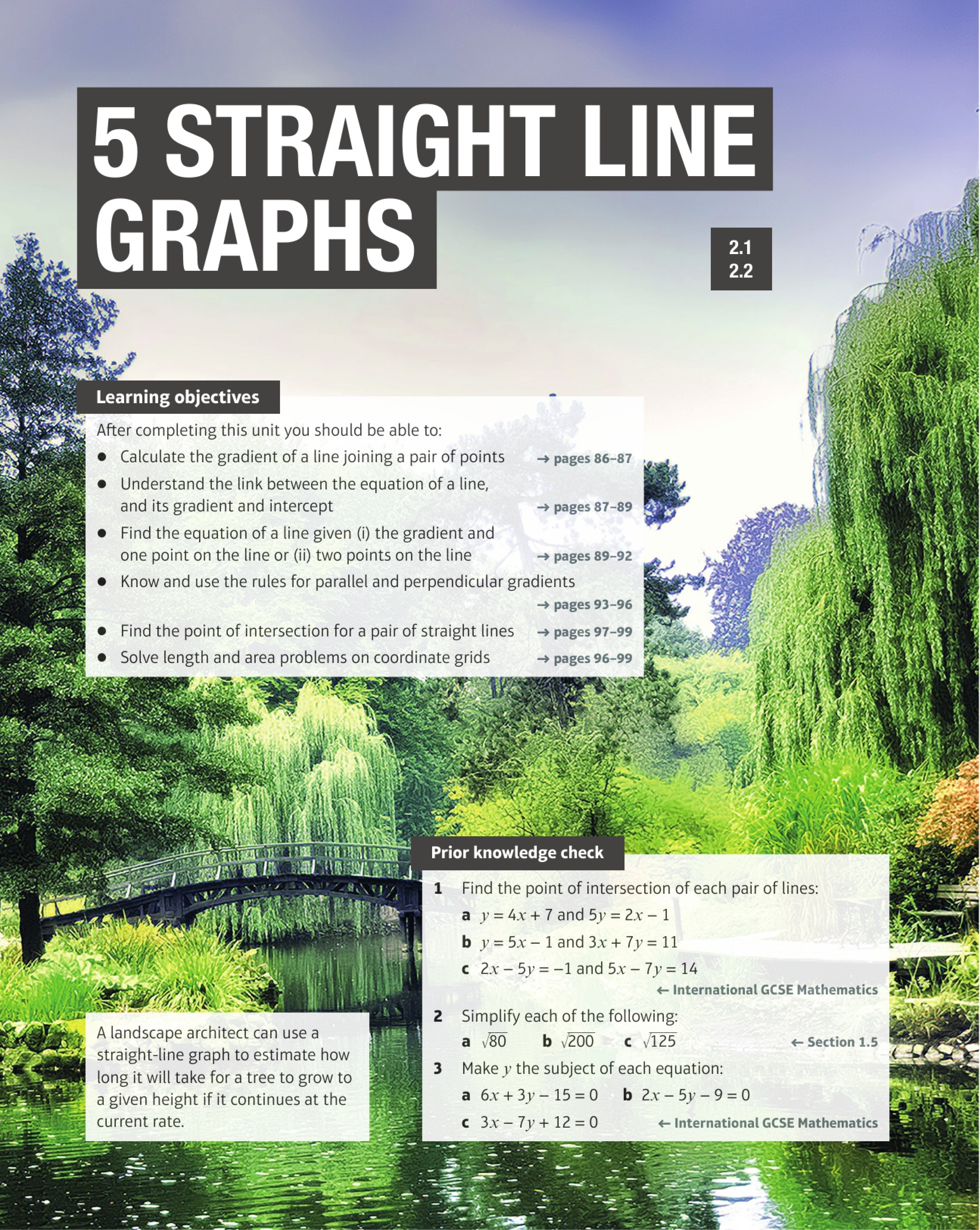
Challenge SKILLS CREATIVITY, INNOVATION

1 **a** Solve the equation $x^2 - 10x + 9 = 0$.

b Hence, or otherwise, solve the equation $3^{x-2}(3^x - 10) = -1$. ← Sections 1.1, 1.3, 2.1

2 A rectangle has an area of 6 cm^2 and a perimeter of $8\sqrt{2}$ cm. Find the dimensions of the rectangle, giving your answers as surds in their simplest form. ← Sections 1.5, 2.2

3 Show algebraically that the graphs of $y = 3x^3 + x^2 - x$ and $y = 2x(x - 1)(x + 1)$ have only one point of intersection, and find the coordinates of this point. ← Section 3.3



5 STRAIGHT LINE GRAPHS

2.1
2.2

Learning objectives

After completing this unit you should be able to:

- Calculate the gradient of a line joining a pair of points → pages 86–87
- Understand the link between the equation of a line, and its gradient and intercept → pages 87–89
- Find the equation of a line given (i) the gradient and one point on the line or (ii) two points on the line → pages 89–92
- Know and use the rules for parallel and perpendicular gradients → pages 93–96
- Find the point of intersection for a pair of straight lines → pages 97–99
- Solve length and area problems on coordinate grids → pages 96–99

A landscape architect can use a straight-line graph to estimate how long it will take for a tree to grow to a given height if it continues at the current rate.

Prior knowledge check

1 Find the point of intersection of each pair of lines:

a $y = 4x + 7$ and $5y = 2x - 1$

b $y = 5x - 1$ and $3x + 7y = 11$

c $2x - 5y = -1$ and $5x - 7y = 14$

← International GCSE Mathematics

2 Simplify each of the following:

a $\sqrt{80}$ **b** $\sqrt{200}$ **c** $\sqrt{125}$ ← Section 1.5

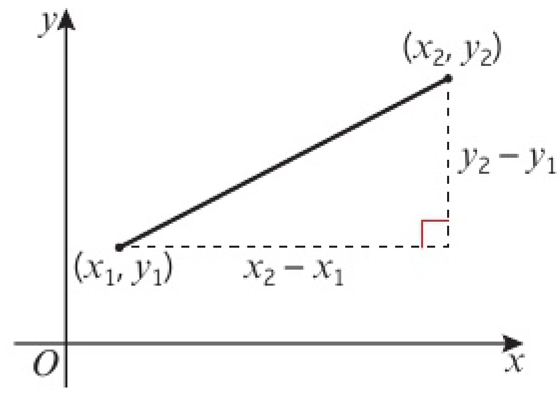
3 Make y the subject of each equation:

a $6x + 3y - 15 = 0$ **b** $2x - 5y - 9 = 0$

c $3x - 7y + 12 = 0$ ← International GCSE Mathematics

5.1 $y = mx + c$

You can find the **gradient** of a straight line joining two points by considering the vertical distance and the horizontal distance between the points.

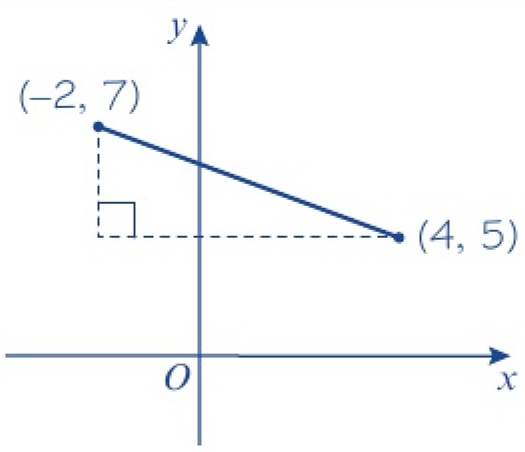


- The gradient m of a line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) can be calculated using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Online Explore the gradient formula using GeoGebra.

Example 1

Work out the gradient of the line joining (−2, 7) and (4, 5).



$m = \frac{5 - 7}{4 - (-2)} = -\frac{2}{6} = -\frac{1}{3}$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$. Here $(x_1, y_1) = (-2, 7)$ and $(x_2, y_2) = (4, 5)$.

Example 2

SKILLS INTERPRETATION

The line joining (2, −5) to (4, a) has gradient −1. Work out the value of a.

$\frac{a - (-5)}{4 - 2} = -1$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$. Here $m = -1$, $(x_1, y_1) = (2, -5)$ and $(x_2, y_2) = (4, a)$.

So $\frac{a + 5}{2} = -1$

$a + 5 = -2$

$a = -7$

Exercise 5A

SKILLS INTERPRETATION

1 Work out the gradients of the lines joining these pairs of points:

a (4, 2), (6, 3)
b (−1, 3), (5, 4)
c (−4, 5), (1, 2)
d (2, −3), (6, 5)
e (−3, 4), (7, −6)
f (−12, 3), (−2, 8)
g (−2, −4), (10, 2)
h $(\frac{1}{2}, 2), (\frac{3}{4}, 4)$
i $(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{2}{3})$
j (−2.4, 9.6), (0, 0)
k (1.3, −2.2), (8.8, −4.7)
l $(0, 5a), (10a, 0)$
m $(3b, -2b), (7b, 2b)$
n $(p, p^2), (q, q^2)$

2 The line joining $(3, -5)$ to $(6, a)$ has gradient 4. Work out the value of a.

3 The line joining $(5, b)$ to $(8, 3)$ has gradient -3. Work out the value of b.

4 The line joining $(c, 4)$ to $(7, 6)$ has gradient $\frac{3}{4}$. Work out the value of c.

5 The line joining $(-1, 2d)$ to $(1, 4)$ has gradient $-\frac{1}{4}$. Work out the value of d.

6 The line joining $(-3, -2)$ to $(2e, 5)$ has gradient 2. Work out the value of e.

7 The line joining $(7, 2)$ to $(f, 3f)$ has gradient 4. Work out the value of f.

8 The line joining $(3, -4)$ to $(-g, 2g)$ has gradient -3. Work out the value of g.

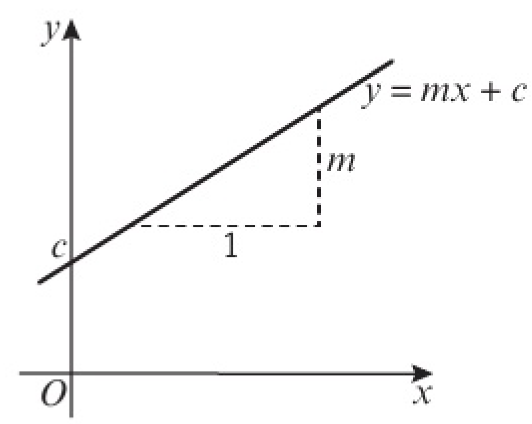
P **9** Show that the points $A(2, 3)$, $B(4, 4)$ and $C(10,7)$ can be joined by a straight line.

Problem-solving Find the gradient of the line joining the points A and B, and the line joining the points A and C.

E/P **10** Show that the points $A(-2a, 5a)$, $B(0, 4a)$ and $C(6a, a)$ are collinear. **(3 marks)**

Notation Points are **collinear** if they all lie on the same straight line.

- The equation of a straight line can be written in the form $y = mx + c$, where m is the gradient and c is the y-intercept.
- The equation of a straight line can also be written in the form $ax + by + c = 0$, where a, b and c are integers.



Example 3

Write down the gradient and y-intercept of these lines:

a $y = -3x + 2$ **b** $4x - 3y + 5 = 0$

a Gradient $= -3$ and y-intercept $= (0, 2)$.

b $y = \frac{4}{3}x + \frac{5}{3}$

Gradient $= \frac{4}{3}$ and y-intercept $= (0, \frac{5}{3})$.

Compare $y = -3x + 2$ with $y = mx + c$. From this, $m = -3$ and $c = 2$.

Rearrange the equation into the form $y = mx + c$. From this, $m = \frac{4}{3}$ and $c = \frac{5}{3}$.

Watch out Use fractions rather than decimals in coordinate geometry questions.

Example 4

Write these lines in the form $ax + by + c = 0$:

a $y = 4x + 3$ **b** $y = -\frac{1}{2}x + 5$

a $4x - y + 3 = 0$

b $\frac{1}{2}x + y - 5 = 0$

$x + 2y - 10 = 0$

Rearrange the equation into the form $ax + by + c = 0$.

Collect all the terms on one side of the equation.

Example 5

SKILLS ANALYSIS

The line $y = 4x - 8$ meets the x-axis at the point P. Work out the coordinates of P.

$4x - 8 = 0$

$4x = 8$

$x = 2$

So P has coordinates (2, 0)

The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 4x - 8$.

Rearrange the equation for x.

Always write down the coordinates of the point.

Exercise 5B

SKILLS ANALYSIS

1 Work out the gradients of these lines:

a $y = -2x + 5$ **b** $y = -x + 7$ **c** $y = 4 + 3x$

d $y = \frac{1}{3}x - 2$ **e** $y = -\frac{2}{3}x$ **f** $y = \frac{5}{4}x + \frac{2}{3}$

g $2x - 4y + 5 = 0$ **h** $10x - 5y + 1 = 0$ **i** $-x + 2y - 4 = 0$

j $-3x + 6y + 7 = 0$ **k** $4x + 2y - 9 = 0$ **l** $9x + 6y + 2 = 0$

2 These lines cut the y-axis at $(0, c)$. Work out the value of c in each case.

a $y = -x + 4$ **b** $y = 2x - 5$ **c** $y = \frac{1}{2}x - \frac{2}{3}$

d $y = -3x$ **e** $y = \frac{6}{7}x + \frac{7}{5}$ **f** $y = 2 - 7x$

g $3x - 4y + 8 = 0$ **h** $4x - 5y - 10 = 0$ **i** $-2x + y - 9 = 0$

j $7x + 4y + 12 = 0$ **k** $7x - 2y + 3 = 0$ **l** $-5x + 4y + 2 = 0$

3 Write these lines in the form $ax + by + c = 0$.

a $y = 4x + 3$ **b** $y = 3x - 2$ **c** $y = -6x + 7$

d $y = \frac{4}{5}x - 6$ **e** $y = \frac{5}{3}x + 2$ **f** $y = \frac{7}{3}x$

g $y = 2x - \frac{4}{7}$ **h** $y = -3x + \frac{2}{9}$ **i** $y = -6x - \frac{2}{3}$

j $y = -\frac{1}{3}x + \frac{1}{2}$ **k** $y = \frac{2}{3}x + \frac{5}{6}$ **l** $y = \frac{3}{5}x + \frac{1}{2}$

4 The line $y = 6x - 18$ meets the x-axis at the point P. Work out the coordinates of P.

5 The line $3x + 2y = 0$ meets the x-axis at the point R. Work out the coordinates of R.

6 The line $5x - 4y + 20 = 0$ meets the y-axis at the point A and the x-axis at the point B. Work out the coordinates of A and B.

7 A line l passes through the points with coordinates (0, 5) and (6, 7).

a Find the gradient of the line.

b Find an equation of the line in the form $ax + by + c = 0$.

E **8** A line l cuts the x-axis at (5, 0) and the y-axis at (0, 2).

a Find the gradient of the line. **(1 mark)**

b Find an equation of the line in the form $ax + by + c = 0$. **(2 marks)**

P **9** Show that the line with equation $ax + by + c = 0$ has gradient $-\frac{a}{b}$ and cuts the y-axis at $-\frac{c}{b}$.

Problem-solving

In question **9**, try solving a similar problem with numbers first:

Find the gradient and y-intercept of the straight line with equation $3x + 7y + 2 = 0$.

E/P **10** The line l with gradient 3 and y-intercept (0, 5) has the equation $ax - 2y + c = 0$. Find the values of a and c. **(2 marks)**

E/P **11** The straight line l passes through (0, 6) and has gradient -2. It intersects the line with equation $5x - 8y - 15 = 0$ at point P. Find the coordinates of P. **(4 marks)**

E/P **12** The straight line l_1 with equation $y = 3x - 7$ intersects the straight line l_2 with equation $ax + 4y - 17 = 0$ at the point $P(-3, b)$.

a Find the value of b. **(1 mark)**

b Find the value of a. **(2 marks)**

Challenge

Show that the equation of a straight line through $(0, a)$ and $(b, 0)$ is $ax + by - ab = 0$.

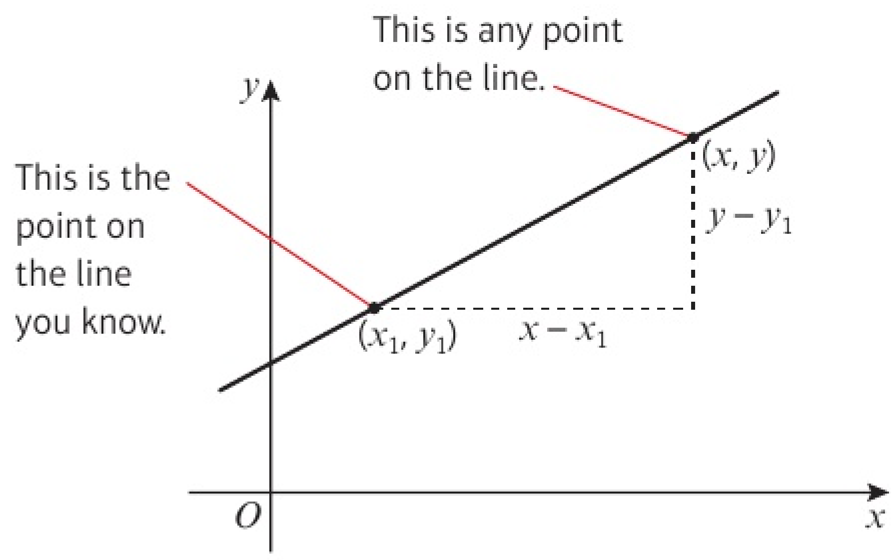
5.2 Equations of straight lines

You can define a straight line by giving:

- one point on the line and the gradient
- two different points on the line

You can find an equation of the line from either of these conditions.

■ The equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) can be written as $y - y_1 = m(x - x_1)$.



Example 6

Find the equation of the line with gradient 5 that passes through the point (3, 2).

$y - 2 = 5(x - 3)$

$y - 2 = 5x - 15$

$y = 5x - 13$

This is in the form $y - y_1 = m(x -$
Here $m = 5$ and $(x_1, y_1) = (3, 2)$.

Online Explore lines of a given
passing through a given point us

Example 7 SKILLS INTERPRETATION

Find the equation of the line that passes through the points (5, 7) and (3, –1).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - (-1)}{5 - 3} = \frac{8}{2} = 4$

So $y - y_1 = m(x - x_1)$

$y + 1 = 4(x - 3)$

$y + 1 = 4x - 12$

$y = 4x - 13$

First find the slope of the line.
Here $(x_1, y_1) = (3, -1)$ and (x_2, y_2)
(x_1, y_1) and (x_2, y_2) have been cho
denominators positive.

You know the gradient and a poi
so use $y - y_1 = m(x - x_1)$.
Use $m = 4$, $x_1 = 3$ and $y_1 = -1$.

Exercise 5C SKILLS INTERPRETATION

1 Find the equation of the line with gradient m that passes through the point $(x_1,$

a $m = 2$ and $(x_1, y_1) = (2, 5)$

b $m = 3$ and $(x_1, y_1) = (-2, 1)$

c $m = -1$ and $(x_1, y_1) = (3, -6)$

d $m = -4$ and $(x_1, y_1) = (-2, -3)$

e $m = \frac{1}{2}$ and $(x_1, y_1) = (-4, 10)$

f $m = -\frac{2}{3}$ and $(x_1, y_1) = (-6, -1)$

g $m = 2$ and $(x_1, y_1) = (a, 2a)$

h $m = -\frac{1}{2}$ and $(x_1, y_1) = (-2b, 3b)$

2 Find the equations of the lines that pass through these pairs of points:

a (2, 4) and (3, 8)

b (0, 2) and (3, 5)

c (–2, 0) and (2, 8)

d (5, –3) and (7, 5)

e (3, –1) and (7, 3)

f (–4, –1) and (6, 4)

g (–1, –5) and (–3, 3)

h (–4, –1) and (–3, –9)

i $(\frac{1}{3}, \frac{2}{5})$ and $(\frac{2}{3}, \frac{4}{5})$

j $(-\frac{3}{4}, \frac{1}{7})$ and $(\frac{1}{4}, \frac{3}{7})$

Hint
fin
th
y -

(E) 3 Find the equation of the line l which passes through the points $A(7, 2)$ and $B(9,$
Give your answer in the form $ax + by + c = 0$.

4 The vertices of the triangle ABC have coordinates $A(3, 5)$, $B(-2, 0)$ and $C(4, -1)$
Find the equations of the sides of the triangle.

E/P **5** The straight line l passes through $(a, 4)$ and $(3a, 3)$. An equation of l is $x + 6y + c = 0$.
Find the value of a and the value of c. **(3 marks)**

E/P **6** The straight line l passes through $(7a, 5)$ and $(3a, 3)$.
An equation of l is $x + by - 12 = 0$.
Find the value of a and the value of b. **(3 marks)**

Problem-solving

It is often easier to find unknown values in the order they are given in the question. Find the value of a first, then find the value of c.

Challenge

Consider the line passing through points (x_1, y_1) and (x_2, y_2).

a Write down the formula for the gradient, m, of the line.

b Show that the general equation of the line can be written in the form $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$.

c Use the equation from part **b** to find an equation of the line passing through the points (–8, 4) and (–1, 7).

Example 8

The line $y = 3x - 9$ meets the x-axis at the point A. Find the equation of the line with gradient $\frac{2}{3}$ that passes through point A. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$0 = 3x - 9$ so $x = 3$. A is the point $(3, 0)$.

$$y - 0 = \frac{2}{3}(x - 3)$$

$$3y = 2x - 6$$

$$-2x + 3y + 6 = 0$$

Online Plot the solution on a graph using technology.

The line meets the x-axis when $y = 0$, so substitute $y = 0$ into $y = 3x - 9$.

Use $y - y_1 = m(x - x_1)$. Here $m = \frac{2}{3}$ and $(x_1, y_1) = (3, 0)$.

Rearrange the equation into the form $ax + by + c = 0$.

Example 9

SKILLS ANALYSIS

The lines $y = 4x - 7$ and $2x + 3y - 21 = 0$ intersect at the point A. The point B has coordinates (–2, 8). Find the equation of the line that passes through points A and B. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

$$2x + 3(4x - 7) - 21 = 0$$

$$2x + 12x - 21 - 21 = 0$$

$$14x = 42$$

$$x = 3$$

$y = 4(3) - 7 = 5$ so A is the point $(3, 5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 5}{-2 - 3} = \frac{3}{-5} = -\frac{3}{5}$$

$$y - 5 = -\frac{3}{5}(x - 3)$$

$$5y - 25 = -3x + 9$$

$$3x + 5y - 34 = 0$$

Online Check solutions to simultaneous equations using your calculator.

Solve the equations simultaneously to find point A. Substitute $y = 4x - 7$ into $2x + 3y - 21 = 0$.

Find the slope of the line connecting A and B.

Use $y - y_1 = m(x - x_1)$ with $m = -\frac{3}{5}$ and $(x_1, y_1) = (3, 5)$.

Exercise 5D

SKILLS ANALYSIS

1 The line $y = 4x - 8$ meets the x-axis at the point A.
Find the equation of the line with gradient 3 that passes through the point A.

2 The line $y = -2x + 8$ meets the y-axis at the point B.
Find the equation of the line with gradient 2 that passes through the point B.

3 The line $y = \frac{1}{2}x + 6$ meets the x-axis at the point C. Find the equation of the line with gradient $\frac{2}{3}$ that passes through the point C. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) 4 The line $y = \frac{1}{4}x + 2$ meets the y-axis at the point B. The point C has coordinates $(-5, 3)$.
Find the gradient of the line joining the points B and C.

(P) 5 The line that passes through the points $(2, -5)$ and $(-7, 4)$ meets the x-axis at the point P. Work out the coordinates of the point P.

Problem-solving

A sketch can help you check whether or not your answer looks correct.

(P) 6 The line that passes through the points $(-3, -5)$ and $(4, 9)$ meets the y-axis at the point G.
Work out the coordinates of the point G.

(P) 7 The line that passes through the points $(3, 2\frac{1}{2})$ and $(-1\frac{1}{2}, 4)$ meets the y-axis at the point J.
Work out the coordinates of the point J.

(P) 8 The lines $y = x$ and $y = 2x - 5$ intersect at the point A. Find the equation of the line with gradient $\frac{2}{5}$ that passes through the point A.

(P) 9 The lines $y = 4x - 10$ and $y = x - 1$ intersect at the point T. Find the equation of the line with gradient $-\frac{2}{3}$ that passes through the point T. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) 10 The line p has gradient $\frac{2}{3}$ and passes through the point $(6, -12)$. The line q has gradient -1 and passes through the point $(5, 5)$. The line p meets the y-axis at A and the line q meets the x-axis at B. Work out the gradient of the line joining the points A and B.

(P) 11 The line $y = -2x + 6$ meets the x-axis at the point P. The line $y = \frac{3}{2}x - 4$ meets the y-axis at the point Q. Find the equation of the line joining the points P and Q.

(P) 12 The line $y = 3x - 5$ meets the x-axis at the point M. The line $y = -\frac{2}{3}x + \frac{2}{3}$ meets the y-axis at the point N. Find the equation of the line joining the points M and N. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) 13 The line $y = 2x - 10$ meets the x-axis at the point A. The line $y = -2x + 4$ meets the y-axis at the point B. Find the equation of the line joining the points A and B.

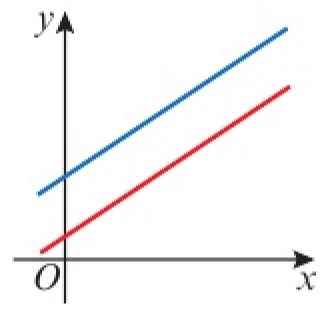
(P) 14 The line $y = 4x + 5$ meets the y-axis at the point C. The line $y = -3x - 15$ meets the x-axis at the point D. Find the equation of the line joining the points C and D. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) 15 The lines $y = x - 5$ and $y = 3x - 13$ intersect at the point S. The point T has coordinates $(-4, 2)$.
Find the equation of the line that passes through the points S and T.

(P) 16 The lines $y = -2x + 1$ and $y = x + 7$ intersect at the point L. The point M has coordinates $(-3, 1)$.
Find the equation of the line that passes through the points L and M.

5.3 Parallel and perpendicular lines

- Parallel lines have the same gradient.



Example 10

A line is parallel to the line $6x + 3y - 2 = 0$ and it passes through the point (0, 3).
Work out the equation of the line.

$6x + 3y - 2 = 0$
$3y - 2 = -6x$
$3y = -6x + 2$
$y = -2x + \frac{2}{3}$
The gradient of this line is −2.
The equation of the line is $y = -2x + 3$.

Rearrange the equation into the form $y = mx + c$ to find m.

Compare $y = -2x + \frac{2}{3}$ with $y = mx + c$, so $m = -2$.
Parallel lines have the same gradient, so the gradient of the required line = −2.

(0, 3) is the intercept on the y-axis, so $c = 3$.

Exercise 5E

SKILLS ANALYSIS

1 Work out whether or not each pair of lines is parallel.

a $y = 5x - 2$
$15x - 3y + 9 = 0$

b $7x + 14y - 1 = 0$
$y = \frac{1}{2}x + 9$

c $4x - 3y - 8 = 0$
$3x - 4y - 8 = 0$

(P) 2 The line r passes through the points (1, 4) and (6, 8) and the line s passes through the points (5, −3) and (20, 9). Show that the lines r and s are parallel.

(P) 3 The coordinates of a **quadrilateral** $ABCD$ are $A(-6, 2)$, $B(4, 8)$, $C(6, 1)$ and $D(-9, -8)$. Show that the quadrilateral is a trapezium.

Hint A trapezium has exactly one pair of parallel sides.

4 A line is parallel to the line $y = 5x + 8$ and its y-intercept is (0, 3). Write down the equation of the line.

Hint The line will have gradient 5.

5 A line is parallel to the line $y = -\frac{2}{5}x + 1$ and its y-intercept is (0, −4). Work out the equation of the line. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

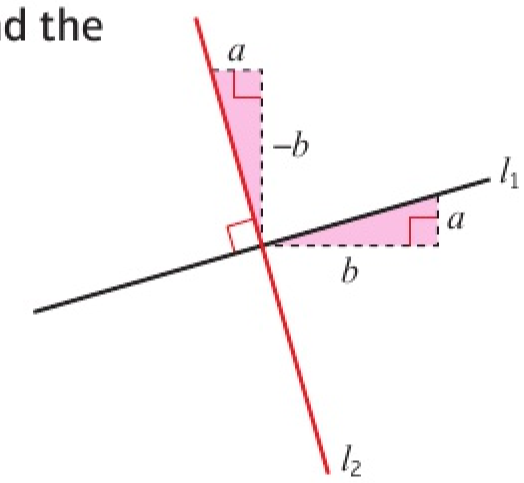
(P) 6 A line is parallel to the line $3x + 6y + 11 = 0$ and its intercept on the y-axis is (0, 7). Write down the equation of the line.

(P) 7 A line is parallel to the line $2x - 3y - 1 = 0$ and it passes through the point (0, 0). Write down the equation of the line.

8 Find an equation of the line that passes through the point (−2, 7) and is parallel to the line $y = 4x + 1$. Write your answer in the form $ax + by + c = 0$.

Perpendicular lines are at right angles to each other. If you know the gradient of one line, you can find the gradient of the other.

- If a line has a gradient of m, a line perpendicular to it has a gradient of $-\frac{1}{m}$
- If two lines are perpendicular, the product of their gradients is −1.



The shaded triangles are **congruent**.

Line l_1 has gradient $\frac{a}{b} = m$

Line l_2 has gradient $\frac{-b}{a} = -\frac{1}{m}$

Example 11

Work out whether these pairs of lines are parallel, perpendicular or neither:

a $3x - y - 2 = 0$
$x + 3y - 6 = 0$

b $y = \frac{1}{2}x$
$2x - y + 4 = 0$

a $3x - y - 2 = 0$ — Rearrange the equations into the form $y = mx + c$.
$3x - 2 = y$
So $y = 3x - 2$
The gradient of this line is 3.

$x + 3y - 6 = 0$
$3y - 6 = -x$
$3y = -x + 6$
$y = -\frac{1}{3}x + 2$
The gradient of this line is $-\frac{1}{3}$ — Compare $y = -\frac{1}{3}x + 2$ with $y = mx + c$, so $m = -\frac{1}{3}$.

So the lines are perpendicular as $3 \times (-\frac{1}{3}) = -1$.

b $y = \frac{1}{2}x$ — Compare $y = \frac{1}{2}x$ with $y = mx + c$, so $m = \frac{1}{2}$.
The gradient of this line is $\frac{1}{2}$

$2x - y + 4 = 0$ — Rearrange the equation into the form $y = mx + c$ to find m.
$2x + 4 = y$
So $y = 2x + 4$
The gradient of this line is 2. — Compare $y = 2x + 4$ with $y = mx + c$, so $m = 2$.

The lines are not parallel as they have different gradients.
The lines are not perpendicular as $\frac{1}{2} \times 2 \neq -1$.

Online Explore this solution using technology.

Example 12

A line is perpendicular to the line $2y - x - 8 = 0$ and passes through the point (5, –7). Find the equation of the line.

Rearranging, $y = \frac{1}{2}x + 4$

Gradient of $y = \frac{1}{2}x + 4$ is $\frac{1}{2}$

So the gradient of the perpendicular line is –2.

$y - y_1 = m(x - x_1)$

$y + 7 = -2(x - 5)$

$y + 7 = -2x + 10$

$y = -2x + 3$

Problem-solving

Fill in the steps of this problem yourself:

- Rearrange the equation into the form $y = mx + c$ to find the gradient.
- Use $-\frac{1}{m}$ to find the gradient of a perpendicular line.
- Use $y - y_1 = m(x - x_1)$ to find the equation of the line.

Exercise 5F

SKILLS ANALYSIS

1 Work out whether these pairs of lines are parallel, perpendicular or neither:

a $y = 4x + 2$
$y = -\frac{1}{4}x - 7$

b $y = \frac{2}{3}x - 1$
$y = \frac{2}{3}x - 11$

c $y = \frac{1}{5}x + 9$
$y = 5x + 9$

d $y = -3x + 2$
$y = \frac{1}{3}x - 7$

e $y = \frac{3}{5}x + 4$
$y = -\frac{5}{3}x - 1$

f $y = \frac{5}{7}x$
$y = \frac{5}{7}x - 3$

g $y = 5x - 3$
$5x - y + 4 = 0$

h $5x - y - 1 = 0$
$y = -\frac{1}{5}x$

i $y = -\frac{3}{2}x + 8$
$2x - 3y - 9 = 0$

j $4x - 5y + 1 = 0$
$8x - 10y - 2 = 0$

k $3x + 2y - 12 = 0$
$2x + 3y - 6 = 0$

l $5x - y + 2 = 0$
$2x + 10y - 4 = 0$

2 A line is perpendicular to the line $y = 6x - 9$ and passes through the point (0, 1). Find an equation of the line.

(P) **3** A line is perpendicular to the line $3x + 8y - 11 = 0$ and passes through the point (0, –8). Find an equation of the line.

4 Find an equation of the line that passes through the point (6, –2) and is perpendicular to the line $y = 3x + 5$.

5 Find an equation of the line that passes through the point (–2, 5) and is perpendicular to the line $y = 3x + 6$.

(P) **6** Find an equation of the line that passes through the point (3, 4) and is perpendicular to the line $4x - 6y + 7 = 0$.

7 Find an equation of the line that passes through the point (5, –5) and is perpendicular to the line $y = \frac{2}{3}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

8 Find an equation of the line that passes through the point (−2, −3) and is perpendicular to the line $y = -\frac{4}{7}x + 5$. Write your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(P) **9** The line l passes through the points (−3, 0) and (3, −2) and the line n passes through the points (1, 8) and (−1, 2). Show that the lines l and n are perpendicular.

Problem-solving

Don't do more work than you need to. You only need to find the gradients of both lines, not their equations.

(P) **10** The vertices of a quadrilateral $ABCD$ have coordinates $A(-1, 5)$, $B(7, 1)$, $C(5, -3)$ and $D(-3, 1)$. Show that the quadrilateral is a rectangle.

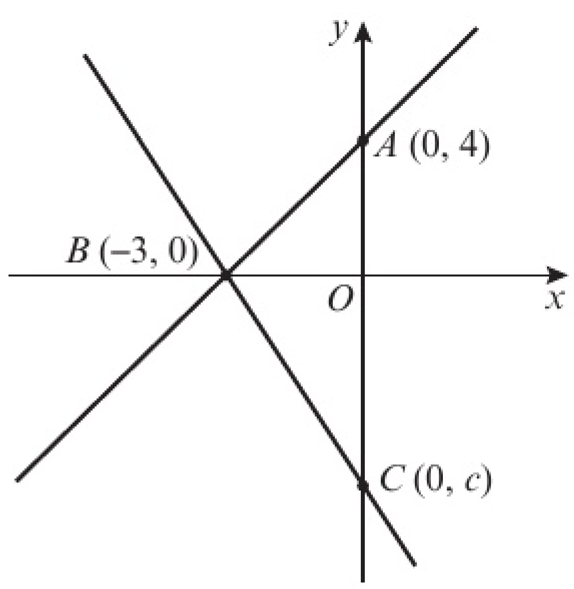
Hint The sides of a rectangle are perpendicular.

(E/P) **11** A line l_1 has equation $5x + 11y - 7 = 0$ and crosses the x-axis at A. The line l_2 is perpendicular to l_1 and passes through A.

a Find the coordinates of the point A. **(1 mark)**

b Find the equation of the line l_2. Write your answer in the form $ax + by + c = 0$. **(3 marks)**

(E/P) **12** The points A and C lie on the y-axis and the point B lies on the x-axis as shown in the diagram.



Problem-solving

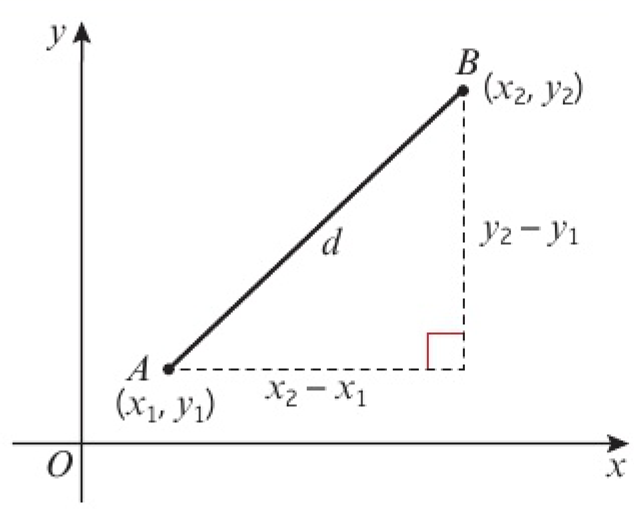
Sketch graphs in coordinate geometry problems are not accurate, but you can use the graph to make sure that your answer makes sense. In this question c must be negative.

The line through points A and B is perpendicular to the line through points B and C. Find the value of c. **(6 marks)**

5.4 Length and area

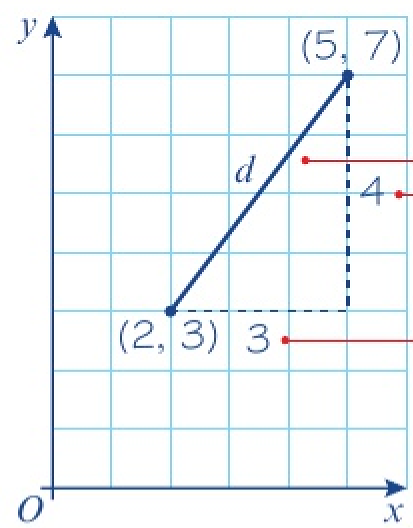
You can find the distance between two points A and B by considering a right-angled triangle with hypotenuse AB.

- You can find the distance d between (x_1, y_1) and (x_2, y_2) by using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Example 13 SKILLS ANALYSIS

Find the distance between (2, 3) and (5, 7).



Draw a sketch.

Let the distance between the points be d.

The difference in the y-coordinates is $7 - 3 = 4$.

The difference in the x-coordinates is $5 - 2 = 3$.

$d^2 = (5 - 2)^2 + (7 - 3)^2$

$d^2 = 3^2 + 4^2$

$d = \sqrt{(3^2 + 4^2)}$

$= \sqrt{25}$

$= 5$

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 7)$.

Example 14

Online Draw both lines and the triangle AOB on a graph using technology.

The straight line l_1 with equation $4x - y = 0$ and the straight line l_2 with equation $2x + 3y - 21 = 0$ intersect at point A.

a Work out the coordinates of A.

b Work out the area of triangle AOB where B is the point where l_2 meets the x-axis.

a Equation of l_1 is $y = 4x$

Rewrite the equation of l_1 in the form $y = mx + c$.

$2x + 3y - 21 = 0$

$2x + 3(4x) - 21 = 0$

Substitute $y = 4x$ into the equation for l_2 to find the point of intersection.

$14x - 21 = 0$

$14x = 21$

$x = \frac{3}{2}$

Solve the equation to find the x-coordinate of point A.

$y = 4 \times \left(\frac{3}{2}\right) = 6$

Substitute to find the y-coordinate of point A.

So point A has coordinates $\left(\frac{3}{2}, 6\right)$.

b The triangle AOB has a height of 6 units.

The height is the y-coordinate of point A.

$2x + 3y - 21 = 0$

$2x + 3(0) - 21 = 0$

B is the point where the line l_2 intersects the x-axis. At B, the y-coordinate is zero.

$2x - 21 = 0$

$x = \frac{21}{2}$

Solve the equation to find the x-coordinate of point B.

The triangle AOB has a base length of $\frac{21}{2}$ units.

Area $= \frac{1}{2} \times 6 \times \frac{21}{2} = \frac{63}{2}$

Area $= \frac{1}{2} \times$ **base** $\times$ height

You don't need to give units for length and area problems on coordinate grids.

Exercise 5G

SKILLS ANALYSIS

1 Find the distance between these pairs of points:

a (0, 1), (6, 9) b (4, −6), (9, 6) c (3, 1), (−1, 4)

d (3, 5), (4, 7) e (0, −4), (5, 5) f (−2, −7), (5, 1)

2 Consider the points $A(-3, 5)$, $B(-2, -2)$ and $C(3, -7)$. Determine whether or not the line joining points A and B is congruent to the line joining points B and C.

Hint Two line segments are congruent if they are the same length.

3 Consider the points $P(11, -8)$, $Q(4, -3)$ and $R(7, 5)$. Show that the **line segment** joining the points P and Q is not congruent to the line joining the points Q and R.

(P) 4 The distance between the points (−1, 13) and $(x, 9)$ is $\sqrt{65}$. Find two possible values of x.

Problem-solving Use the distance formula to formulate a quadratic equation in x.

(P) 5 The distance between the points $(2, y)$ and (5, 7) is $3\sqrt{10}$. Find two possible values of y.

(P) 6 a Show that the straight line l_1 with equation $y = 2x + 4$ is parallel to the straight line l_2 with equation $6x - 3y - 9 = 0$.

b Find the equation of the straight line l_3 that is perpendicular to l_1 and passes through the point (3, 10).

c Find the point of intersection of the lines l_2 and l_3.

d Find the shortest distance between lines l_1 and l_2.

Problem-solving The shortest distance between two parallel lines is the perpendicular distance between them.

(E/P) 7 A point P lies on the line with equation $y = 4 - 3x$. The point P is a distance $\sqrt{34}$ from the origin. Find the two possible positions of point P. **(5 marks)**

(P) 8 The vertices of a triangle are $A(2, 7)$, $B(5, -6)$ and $C(8, -6)$.

a Show that the triangle is a **scalene** triangle.

b Find the area of the triangle ABC.

Notation Scalene triangles have three sides of different lengths.

Problem-solving Draw a sketch and label the points A, B and C. Find the length of the base and the height of the triangle.

9 The straight line l_1 has equation $y = 7x - 3$. The straight line l_2 has equation $4x + 3y - 41 = 0$. The lines intersect at the point A.

a Work out the coordinates of A.

The straight line l_2 crosses the x-axis at the point B.

b Work out the coordinates of B.

c Work out the area of triangle AOB, where O is the origin.

10 The straight line l_1 has equation $4x - 5y - 10 = 0$ and intersects the x-axis at point A.
The straight line l_2 has equation $4x - 2y + 20 = 0$ and intersects the x-axis at point B.

a Work out the coordinates of A.

b Work out the coordinates of B.

The straight lines l_1 and l_2 intersect at the point C.

c Work out the coordinates of C.

d Work out the area of triangle ABC.

E 11 The points $R(5, -2)$ and $S(9, 0)$ lie on the straight line l_1 as shown.

a Work out an equation for straight line l_1. **(2 marks)**

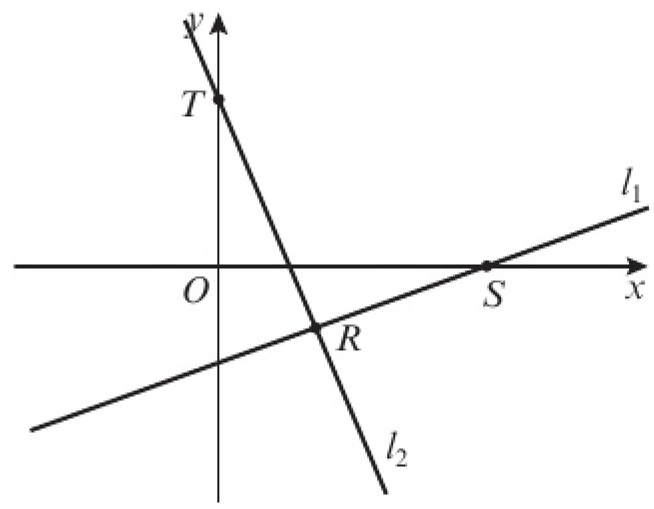
The straight line l_2 is perpendicular to l_1 and passes through the point R.

b Work out an equation for straight line l_2. **(2 marks)**

c Write down the coordinates of T. **(1 mark)**

d Work out the lengths of RS and TR leaving your answer in the form $k\sqrt{5}$. **(2 marks)**

e Work out the area of $\triangle RST$. **(2 marks)**



E/P 12 The straight line l_1 passes through the point $(-4, 14)$ and has gradient $-\frac{1}{4}$.

a Find an equation for l_1 in the form $ax + by + c = 0$, where a, b and c are integers. **(3 marks)**

b Write down the coordinates of A, the point where straight line l_1 crosses the y-axis. **(1 mark)**

The straight line l_2 passes through the origin and has gradient 3.
The lines l_1 and l_2 intersect at the point B.

c Calculate the coordinates of B. **(2 marks)**

d Calculate the exact area of $\triangle OAB$. **(2 marks)**

Chapter review 5

SKILLS EXECUTIVE FUNCTION

E/P 1 The straight line passing through the point $P(2, 1)$ and the point $Q(k, 11)$ has gradient $-\frac{5}{12}$.

a Find the equation of the line in terms of x and y only. **(2 marks)**

b Determine the value of k. **(2 marks)**

E/P 2 The points A and B have coordinates $(k, 1)$ and $(8, 2k - 1)$ respectively, where k is a constant.
Given that the gradient of AB is $\frac{1}{3}$

a show that $k = 2$ **(2 marks)**

b find an equation for the line through A and B. **(3 marks)**

E 3 The line L_1 has gradient $\frac{1}{7}$ and passes through the point $A(2, 2)$. The line L_2 has gradient -1 and passes through the point $B(4, 8)$. The lines L_1 and L_2 intersect at the point C.

a Find an equation for L_1 and an equation for L_2. **(4 marks)**

b Determine the coordinates of C. **(2 marks)**

(E) **4** **a** Find an equation of the line l which passes through the points $A(1, 0)$ and $B(5, 6)$. **(2 marks)**

The line m with equation $2x + 3y = 15$ meets l at the point C.

b Determine the coordinates of C. **(2 marks)**

(E) **5** The line L passes through the points $A(1, 3)$ and $B(-19, -19)$.
Find an equation of L in the form $ax + by + c = 0$. where a, b and c are integers. **(3 marks)**

(E) **6** The straight line l_1 passes through the points A and B with coordinates $(2, 2)$ and $(6, 0)$ respectively.

a Find an equation of l_1. **(3 marks)**

The straight line l_2 passes through the point C with coordinate $(-9, 0)$ and has gradient $\frac{1}{4}$.

b Find an equation of l_2. **(2 marks)**

(E/P) **7** The straight line l passes through $A(1, 3\sqrt{3})$ and $B(2 + \sqrt{3}, 3 + 4\sqrt{3})$.
Show that l meets the x-axis at the point $C(-2, 0)$. **(5 marks)**

(E) **8** The points A and B have coordinates $(-4, 6)$ and $(2, 8)$ respectively.
A line p is drawn through B perpendicular to AB to meet the y-axis at the point C.

a Find an equation of the line p. **(3 marks)**

b Determine the coordinates of C. **(1 mark)**

(E/P) **9** The line l has equation $2x - y - 1 = 0$.
The line m passes through the point $A(0, 4)$ and is perpendicular to the line l.

a Find an equation of m. **(2 marks)**

b Show that the lines l and m intersect at the point $P(2, 3)$. **(2 marks)**

The line n passes through the point $B(3, 0)$ and is parallel to the line m.

c Find the coordinates of the point of intersection of the lines l and n. **(3 marks)**

(E/P) **10** The line l_1 passes through the points A and B with coordinates $(0, -2)$ and $(6, 7)$ respectively. The line l_2 has equation $x + y = 8$ and cuts the y-axis at the point C.
The line l_1 and l_2 intersect at D.
Find the area of triangle ACD. **(6 marks)**

(E) **11** The points A and B have coordinates $(2, 16)$ and $(12, -4)$ respectively.
A straight line l_1 passes through A and B.

a Find an equation for l_1 in the form $ax + by = c$. **(2 marks)**

The line l_2 passes through the point C with coordinates $(-1, 1)$ and has gradient $\frac{1}{3}$.

b Find an equation for l_2. **(2 marks)**

E/P **12** The points $A(-1, -2)$, $B(7, 2)$ and $C(k, 4)$, where k is a constant, are the vertices of $\triangle ABC$. Angle ABC is a right angle.

a Find the gradient of AB. **(1 mark)**

b Calculate the value of k. **(2 marks)**

c Find an equation of the straight line passing through B and C. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(2 marks)**

d Calculate the area of $\triangle ABC$. **(2 marks)**

E/P **13** **a** Find an equation of the straight line passing through the points with coordinates $(-1, 5)$ and $(4, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(3 marks)**

The line crosses the x-axis at the point A and the y-axis at the point B, and O is the origin.

b Find the area of $\triangle AOB$. **(3 marks)**

E **14** The straight line l_1 has equation $4y + x = 0$.

The straight line l_2 has equation $y = 2x - 3$.

a On the same axes, sketch the graphs of l_1 and l_2. Show clearly the coordinates of all points at which the graphs meet the coordinate axes. **(2 marks)**

The lines l_1 and l_2 intersect at the point A.

b Calculate, as exact fractions, the coordinates of A. **(2 marks)**

c Find an equation of the line through A which is perpendicular to l_1. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(2 marks)**

E **15** The points A and B have coordinates $(4, 6)$ and $(12, 2)$ respectively. The straight line l_1 passes through A and B.

a Find an equation for l_1 in the form $ax + by + c = 0$, where a, b and c are integers. **(3 marks)**

The straight line l_2 passes through the origin and has gradient $-\frac{2}{3}$.

b Write down an equation for l_2. **(1 mark)**

The lines l_1 and l_2 intersect at the point C.

c Find the coordinates of C. **(2 marks)**

d Show that the lines OA and OC are perpendicular, where O is the origin. **(2 marks)**

e Work out the lengths of OA and OC. Write your answers in the form $k\sqrt{13}$. **(2 marks)**

f Hence calculate the area of $\triangle OAC$. **(2 marks)**

16 **a** Use the distance formula to find the distance between $(4a, a)$ and $(-3a, 2a)$.

Hence find the distance between the following pairs of points:

b $(4, 1)$ and $(-3, 2)$ **c** $(12, 3)$ and $(-9, 6)$ **d** $(-20, -5)$ and $(15, -10)$

E/P **17** A is the point $(-1, 5)$. Let (x, y) be any point on the line $y = 3x$.

a Write an equation in terms of x for the distance between (x, y) and $A(-1, 5)$. **(3 marks)**

b Find the coordinates of the two points, B and C, on the line $y = 3x$ which are a distance of $\sqrt{74}$ from $(-1, 5)$. **(3 marks)**

c Find the equation of the line l_1 that is perpendicular to $y = 3x$ and goes through the point $(-1, 5)$. **(2 marks)**

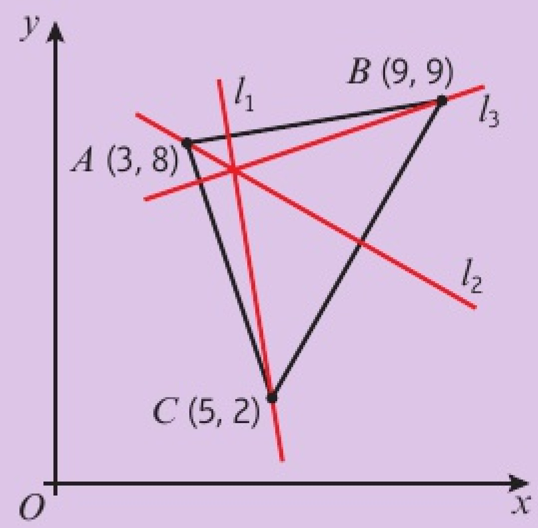
d Find the coordinates of the point of intersection between l_1 and $y = 3x$. **(2 marks)**

e Find the area of triangle ABC. **(2 marks)**

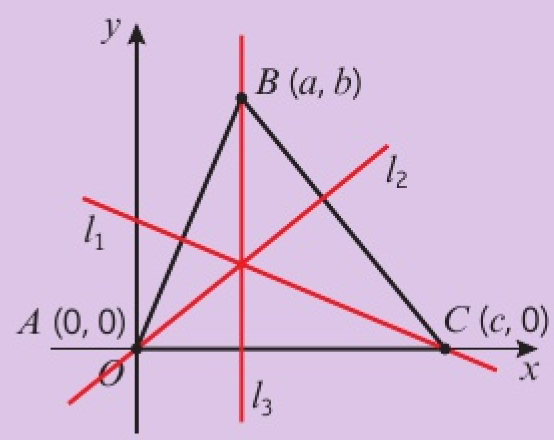
Challenge

1 Find the area of the triangle with vertices $A(-2, -2)$, $B(13, 8)$ and $C(-4, 14)$.

2 A triangle has vertices $A(3, 8)$, $B(9, 9)$ and $C(5, 2)$ as shown in the diagram.
The line l_1 is perpendicular to AB and passes through C.
The line l_2 is perpendicular to BC and passes through A.
The line l_3 is perpendicular to AC and passes through B.
Show that the lines l_1, l_2 and l_3 meet at a point and find the coordinates of that point.



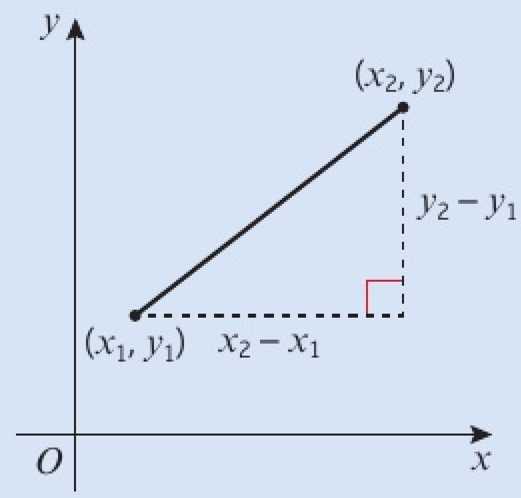
3 A triangle has vertices $A(0, 0)$, $B(a, b)$ and $C(c, 0)$ as shown in the diagram.
The line l_1 is perpendicular to AB and passes through C.
The line l_2 is perpendicular to BC and passes through A.
The line l_3 is perpendicular to AC and passes through B.
Find the coordinates of the point of intersection of l_1, l_2 and l_3.



Summary of key points

1 The gradient m of the line joining the point with coordinates (x_1, y_1) to the point with coordinates (x_2, y_2) can be calculated using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



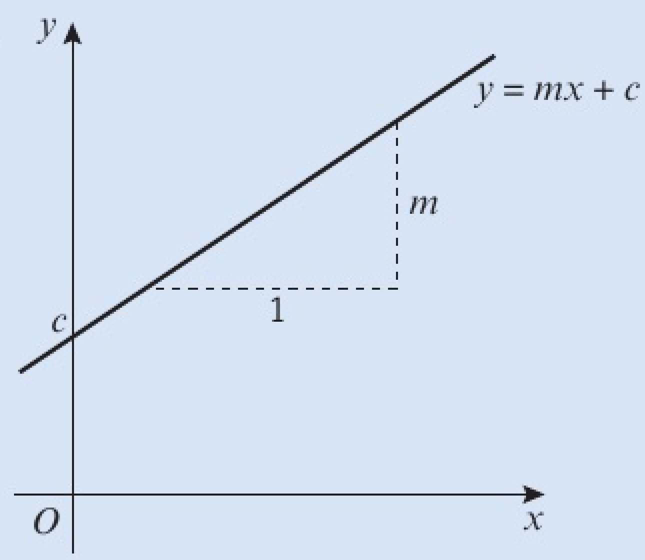
2
- The equation of a straight line can be written in the form

  $y = mx + c$,

  where m is the gradient and $(0, c)$ is the y-intercept.
- The equation of a straight line can also be written in the form

  $ax + by + c = 0$,

  where a, b and c are integers.



3 The equation of a line with gradient m that passes through the point with coordinates (x_1, y_1) can be written as $y - y_1 = m(x - x_1)$.

4 Parallel lines have the same gradient.

5 If a line has a gradient m, a line perpendicular to it has a gradient of $-\frac{1}{m}$.

6 If two lines are perpendicular, the product of their gradients is -1.

7 You can find the distance d between (x_1, y_1) and (x_2, y_2) by using the formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

8 The point of intersection of two lines can be found using simultaneous equations.

6 TRIGONOMETRIC RATIOS

1.11
1.12
3.1
3.3

Learning objectives

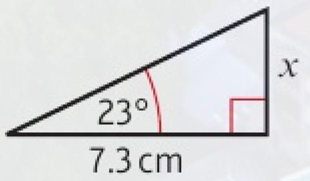
After completing this chapter you should be able to:

- Use the cosine rule to find a missing side or angle → pages 105–110
- Use the sine rule to find a missing side or angle → pages 110–116
- Find the area of a triangle using an appropriate formula → pages 116–118
- Solve problems involving triangles → pages 118–122
- Sketch the graphs of the sine, cosine and tangent functions → pages 123–125
- Sketch simple transformations of these graphs → pages 125–129

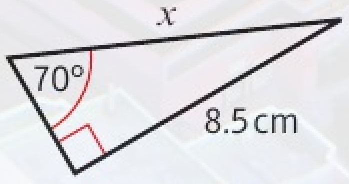
Prior knowledge check

1 Use trigonometry to find the lengths of the marked sides.

a



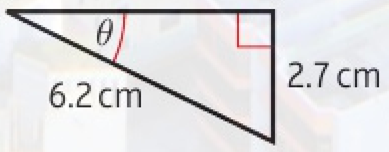
b



← International GCSE Mathematics

2 Find the sizes of the angles marked.

a



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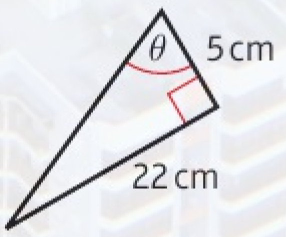


← International GCSE Mathematics

3 $f(x) = x^2 + 3x$. Sketch the graphs of

a $y = f(x)$ **b** $y = f(x + 2)$

c $y = f(x) - 3$ **d** $y = f(\frac{1}{2}x)$

← Sections 4.4, 4.5

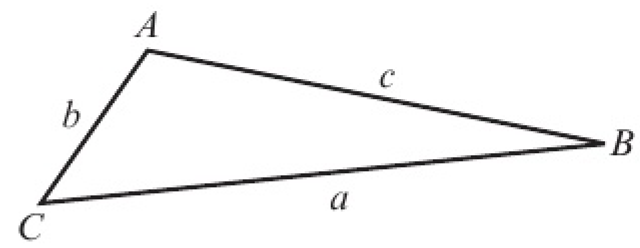
Trigonometry in both two and three dimensions is used by surveyors to work out distances and areas when planning building projects. You will also use trigonometry when working with vector quantities in mechanics.

6.1 The cosine rule

The cosine rule can be used to work out missing sides or angles in triangles.

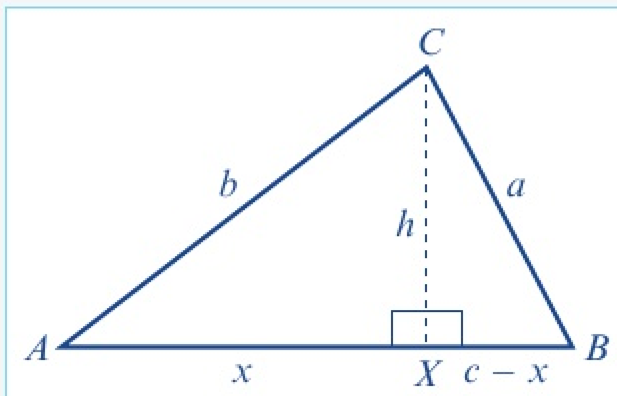
- This version of the cosine rule is used to find a missing side if you know two sides and the angle between them:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



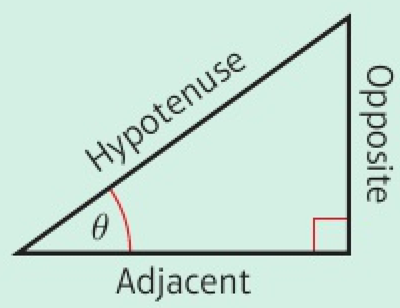
Watch out You can exchange the letters depending on which side you want to find, as long as each side has the same letter as the **opposite** angle.

You can use the standard **trigonometric** ratios for right-angled triangles to prove the cosine rule:



	$h^2 + x^2 = b^2$	Use Pythagoras' theorem in $\triangle CAX$.
and	$h^2 + (c - x)^2 = a^2$	Use Pythagoras' theorem in $\triangle CBX$.
So	$x^2 - (c - x)^2 = b^2 - a^2$	Subtract the two equations.
	$2cx - c^2 = b^2 - a^2$	$(c - x)^2 = c^2 - 2cx + x^2$ So $x^2 - (c - x)^2 = x^2 - c^2 + 2cx - x^2$
	$a^2 = b^2 + c^2 - 2cx$ (1)	Rearrange.
but	$x = b \cos A$ (2)	Use the cosine ratio $\cos A = \frac{x}{b}$ in $\triangle CAX$.
So	$a^2 = b^2 + c^2 - 2bc \cos A$	Combine (1) and (2). This is the cosine rule.

Hint For a right-angled triangle:



$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

If you are given all three sides and asked to find an angle, the cosine rule can be rearranged.

$$a^2 + 2bc \cos A = b^2 + c^2$$

$$2bc \cos A = b^2 + c^2 - a^2$$

Hence $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

You can exchange the letters depending on which angle you want to find.

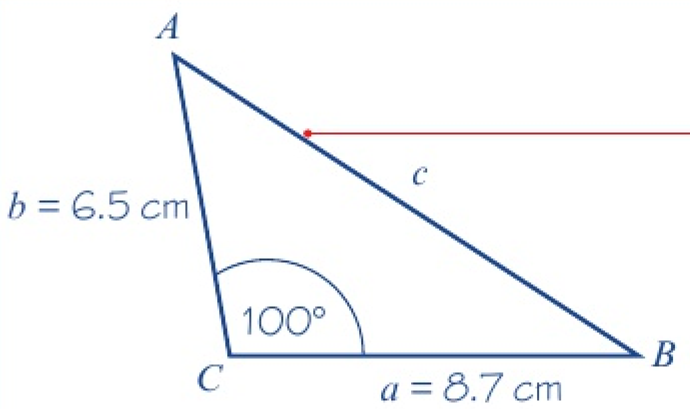
- This version of the cosine rule is used to find an angle if you know all three sides:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Online Explore the cosine rule using technology.

Example 1

Calculate the length of side AB in triangle ABC in which $AC = 6.5$ cm, $BC = 8.7$ cm and $\angle ACB = 100°$.



Label the sides of the triangle with small letters a, b and c opposite the angles marked.

$c^2 = a^2 + b^2 - 2ab\cos C$

Write out the formula you are using as the first line of working, then substitute in the values given.

$c^2 = 8.7^2 + 6.5^2 - 2 \times 8.7 \times 6.5 \times \cos 100°$

$= 75.69 + 42.25 - (-19.639\ldots)$

Don't round any values until the end of your working. You can write your final answer to 3 **significant figures**.

$= 137.57\ldots$

So $c = 11.729\ldots$

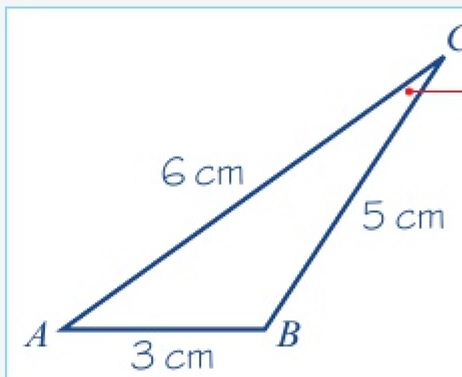
Find the square root.

So $AB = 11.7$ cm (3 s.f.)

Example 2

SKILLS ANALYSIS

Find the size of the smallest angle in a triangle whose sides have lengths 3 cm, 5 cm and 6 cm.



Label the triangle ABC.
The smallest angle is opposite the smallest side so angle C is required.

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Use the cosine rule $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

$= \dfrac{5^2 + 6^2 - 3^2}{2 \times 5 \times 6}$

$= 0.8666\ldots$

Online Use your calculator to work this out efficiently.

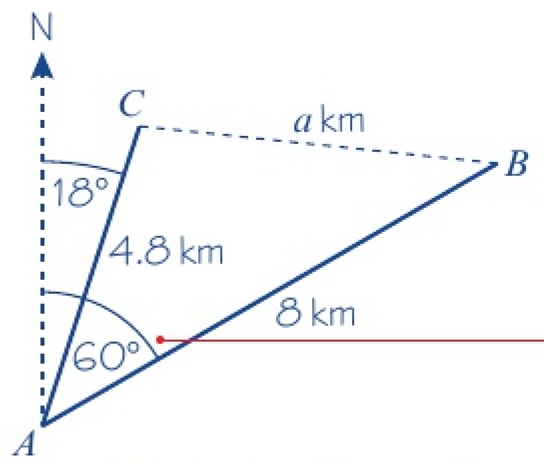
$C = 29.9°$ (3 s.f.)

$C = \cos^{-1}(0.8666\ldots)$

The size of the smallest angle is 29.9°.

Example 3

Coastguard station B is 8 km, on a bearing of 060°, from coastguard station A. A ship C is 4.8 km, on a bearing of 018°, away from A. Calculate how far C is from B.



Problem-solving

If no diagram is given with a question, you should draw one carefully. Double-check that the information given in the question matches your sketch.

In $\triangle ABC$, $\angle CAB = 60° - 18° = 42°$.

$a^2 = b^2 + c^2 - 2bc \cos A$

You now have $b = 4.8$ km, $c = 8$ km and $A = 42°$. Use the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$.

$a^2 = 4.8^2 + 8^2 - 2 \times 4.8 \times 8 \times \cos 42°$

$= 29.966\ldots$

If possible, work this out in one go using your calculator.

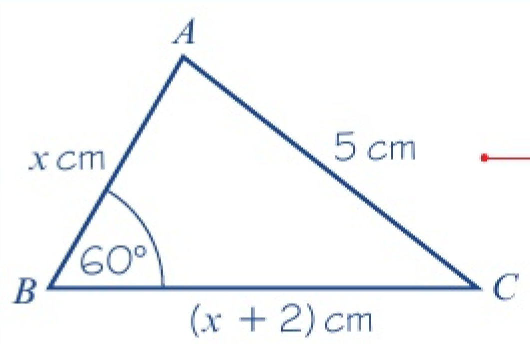
$a = 5.47$ (3 s.f.)

Take the square root of 29.966… and round your final answer to 3 significant figures.

C is 5.47 km from coastguard station B.

Example 4

In $\triangle ABC$, $AB = x$ cm, $BC = (x + 2)$ cm, $AC = 5$ cm and $\angle ABC = 60°$. Find the value of x.



Use the information given in the question to draw a sketch.

$b^2 = a^2 + c^2 - 2ac \cos B$

$5^2 = (x + 2)^2 + x^2 - 2x(x + 2) \cos 60°$

So $25 = 2x^2 + 4x + 4 - x^2 - 2x$

Carefully expand and simplify the right-hand side. Note that $\cos 60° = \frac{1}{2}$.

$x^2 + 2x - 21 = 0$

Rearrange to the form $ax^2 + bx + c = 0$.

$x = \frac{-2 \pm \sqrt{88}}{2}$

Solve the quadratic equation using the quadratic formula. ← Section 2.1

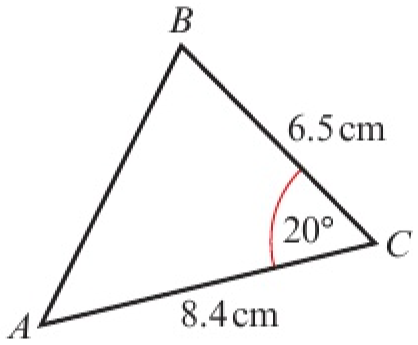
$= 3.69$ cm (3 s.f.)

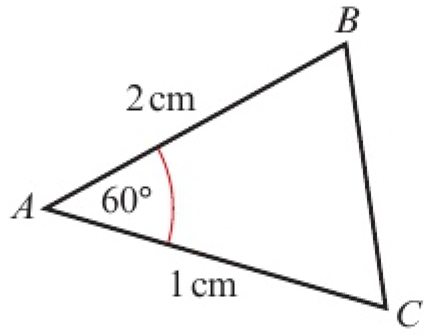
x represents a length so it cannot be negative.

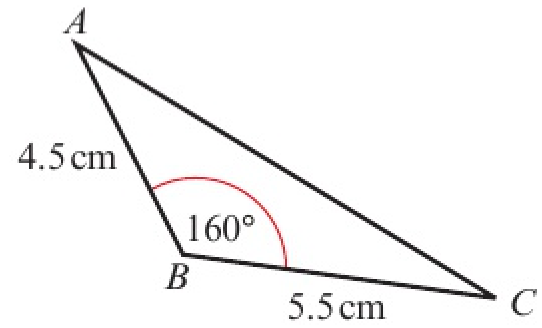
Exercise 6A SKILLS PROBLEM SOLVING

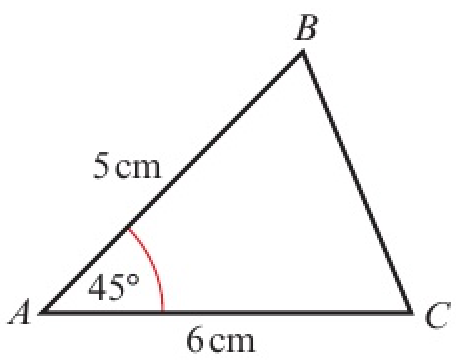
Give answers to 3 significant figures, where appropriate.

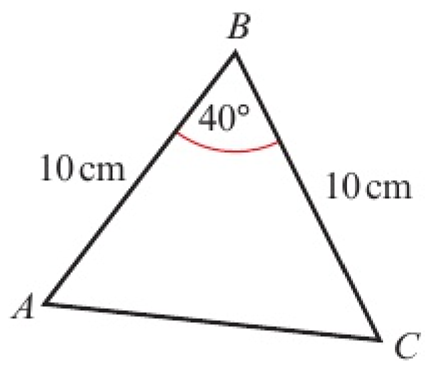
1 In each of the following triangles, calculate the length of the missing side:

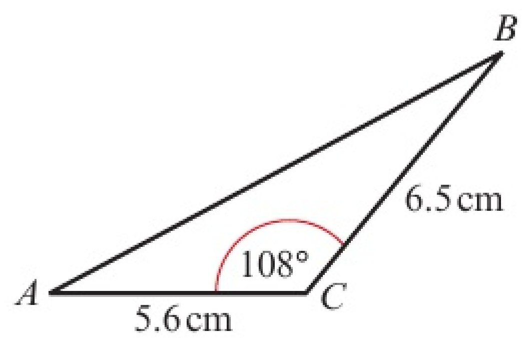
a


b


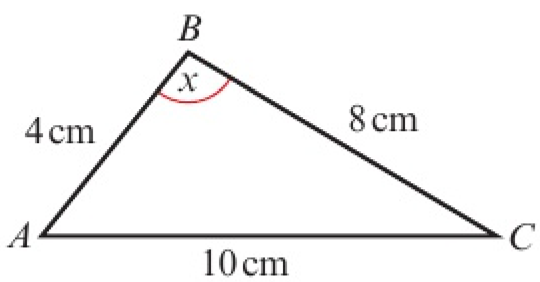
c


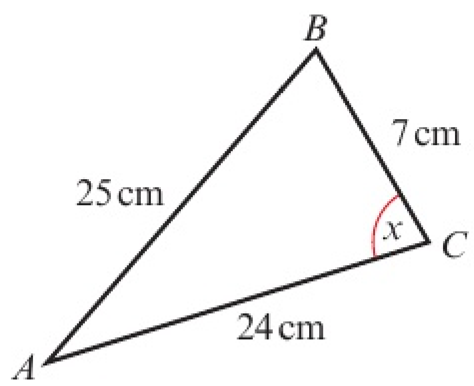
d


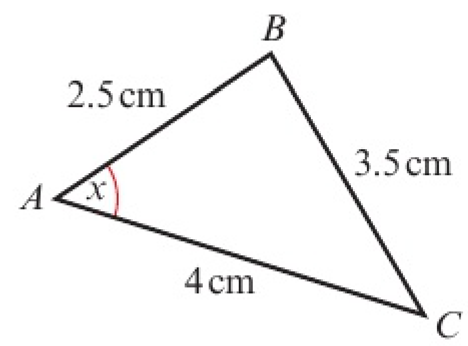
e


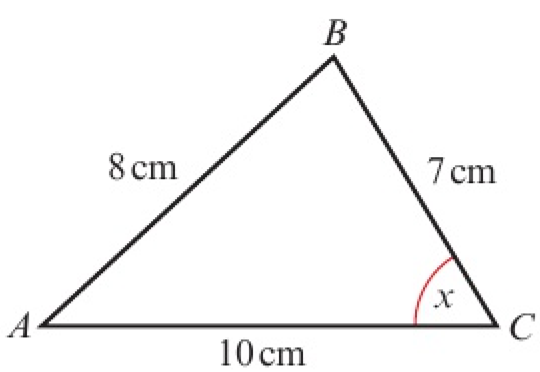
f


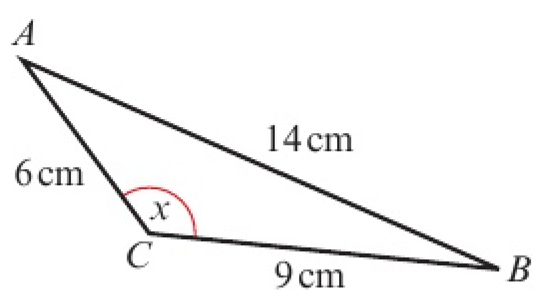
2 In the following triangles, calculate the size of the angle marked x:

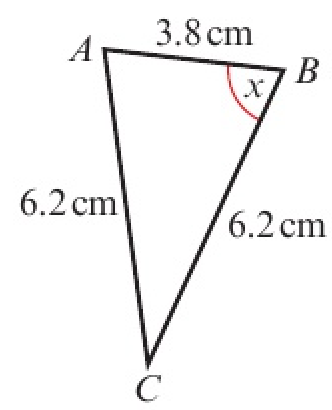
a


b


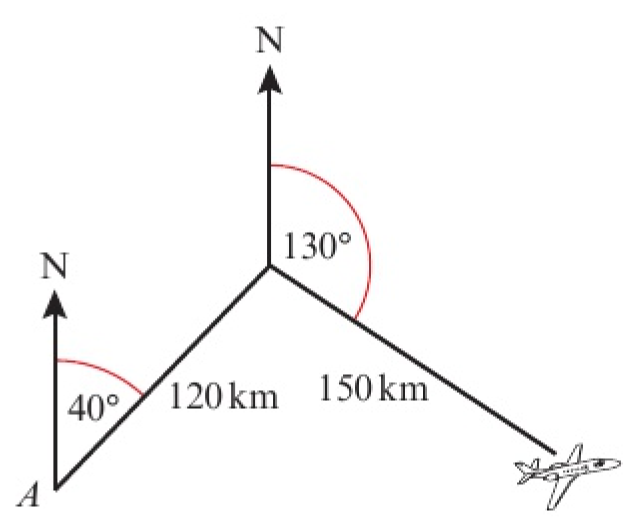
c


d


e


f


3 A plane flies from airport A on a bearing of 040° for 120 km and then on a bearing of 130° for 150 km. Calculate the distance of the plane from the airport.

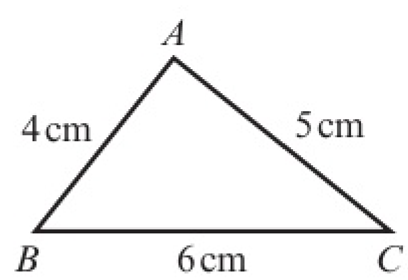


4 From a point A, a boat sails due north for 7 km to B. The boat leaves B and moves on a bearing of 100° for 10 km until it reaches C. Calculate the distance of C from A.

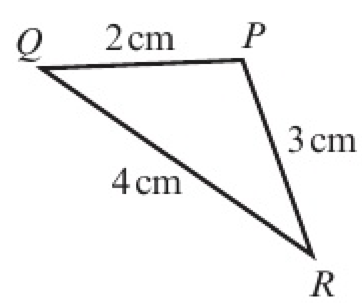
5 A helicopter flies on a bearing of 080° from A to B, where $AB = 50$ km.
It then flies for 60 km to a point C.
Given that C is 80 km from A, calculate the bearing of C from A.

6 The distance from the tee, T, to the flag, F, on a particular hole of a golf course is 494 yards.
A golfer's tee shot travels 220 yards and lands at the point S, where $\angle STF = 22°$.
Calculate how far the ball is from the flag.

(P) 7 Show that $\cos A = \frac{1}{8}$.



(P) 8 Show that $\cos P = -\frac{1}{4}$.



9 In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $AC = 10$ cm.
Calculate the size of the smallest angle.

10 In $\triangle ABC$, $AB = 9.3$ cm, $BC = 6.2$ cm and $AC = 12.7$ cm.
Calculate the size of the largest angle.

(P) 11 The lengths of the sides of a triangle are in the ratio $2:3:4$.
Calculate the size of the largest angle.

12 In $\triangle ABC$, $AB = (x - 3)$ cm, $BC = (x + 3)$ cm, $AC = 8$ cm and $\angle BAC = 60°$.
Use the cosine rule to find the value of x.

(P) 13 In $\triangle ABC$, $AB = x$ cm, $BC = (x - 4)$ cm, $AC = 10$ cm and $\angle BAC = 60°$.
Calculate the value of x.

(P) 14 In $\triangle ABC$, $AB = (5 - x)$ cm, $BC = (4 + x)$ cm, $\angle ABC = 120°$ and $AC = y$ cm.

a Show that $y^2 = x^2 - x + 61$.

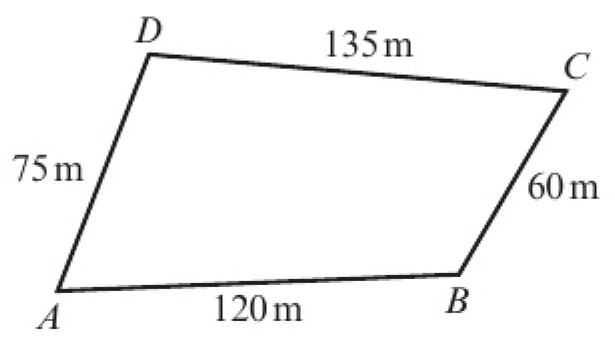
b Use the method of completing the square to find the minimum value of y^2, and give the value of x for which this occurs.

(P) **15** In $\triangle ABC$, $AB = x$ cm, $BC = 5$ cm, $AC = (10 - x)$ cm.

a Show that $\cos \angle ABC = \frac{4x - 15}{2x}$.

b Given that $\cos \angle ABC = -\frac{1}{7}$, work out the value of x.

(P) **16** A farmer has a field in the shape of a quadrilateral as shown.



Problem-solving

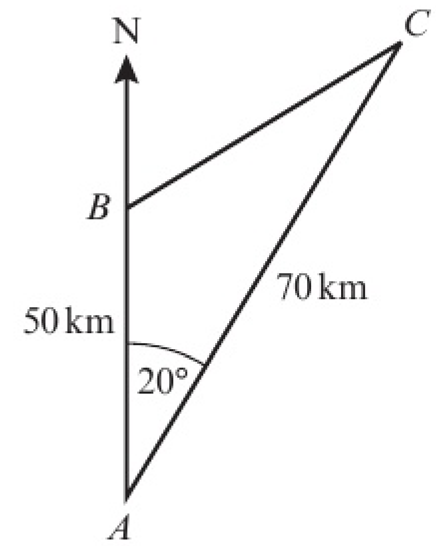
You will have to use the cosine rule twice. Copy the diagram and write any angles or lengths you work out on your copy.

The angle between fences AB and AD is 74°. Find the angle between fences BC and CD.

(E/P) **17** The diagram shows three cargo ships, A, B and C, which are in the same horizontal plane. Ship B is 50 km due north of ship A and ship C is 70 km from ship A. The bearing of C from A is 020°.

a Calculate the distance between ships B and C, in kilometres to 3 s.f. **(3 marks)**

b Calculate the bearing of ship C from ship B. **(4 marks)**

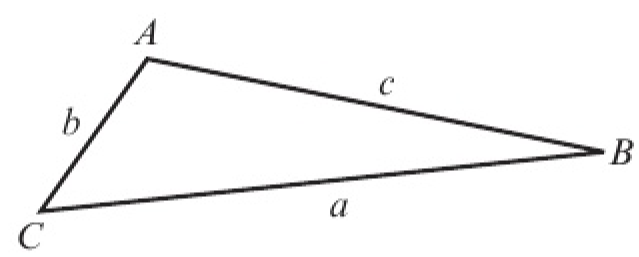


6.2 The sine rule

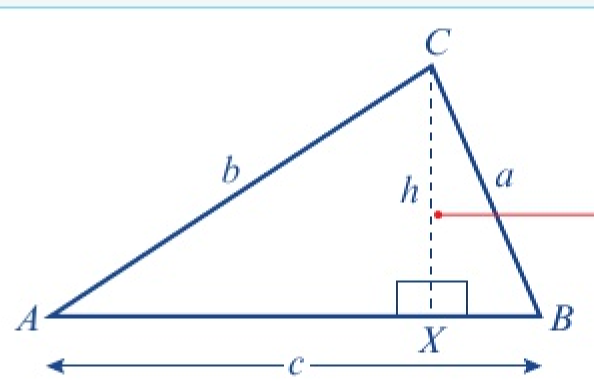
The sine rule can be used to work out missing sides or angles in triangles.

- **This version of the sine rule is used to find the length of a missing side:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



You can use the standard trigonometric ratios for right-angled triangles to prove the sine rule:



$\sin B = \frac{h}{a} \Rightarrow h = a \sin B$

In a general triangle ABC, draw the perpendicular from C to AB. It meets AB at X.

The length of CX is h.

Use the sine ratio in triangle CBX.

Online Explore the sine rule using technology.

and $\sin A = \frac{h}{b} \Rightarrow h = b \sin A$ — Use the sine ratio in triangle CAX.

So $a \sin B = b \sin A$

$\frac{a}{\sin A} = \frac{b}{\sin B}$ — Divide throughout by $\sin A \sin B$.

In a similar way, by drawing the perpendicular from B to the side AC, you can show that:

$\frac{a}{\sin A} = \frac{c}{\sin C}$

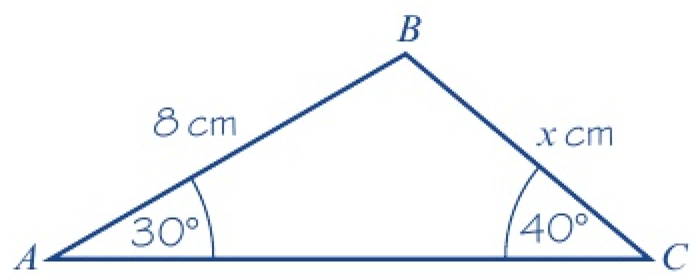
So $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ — This is the sine rule and is true for all triangles.

- This version of the sine rule is used to find a missing angle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example 5

In $\triangle ABC$, $AB = 8$ cm, $\angle BAC = 30°$ and $\angle BCA = 40°$. Find BC.



Always draw a diagram and carefully add the data. Here $c = 8$ cm, $C = 40°$, $A = 30°$ and $a = x$ cm.

In a triangle, the larger a side is, the larger the opposite angle is. Here, as $C > A$, then $c > a$, so you know that $8 > x$.

$\frac{a}{\sin A} = \frac{b}{\sin B}$ — Use this version of the sine rule to find a missing side. Write the formula you are going to use as the first line of working.

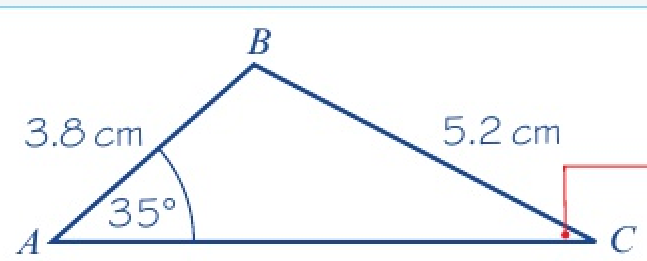
$\frac{x}{\sin 30°} = \frac{8}{\sin 40°}$

So $x = \frac{8 \sin 30°}{\sin 40°} = 6.2228...$ — Multiply throughout by $\sin 30°$.

$= 6.22$ cm (3 s.f.) — Give your answer to 3 significant figures.

Example 6 SKILLS ANALYSIS

In $\triangle ABC$, $AB = 3.8$ cm, $BC = 5.2$ cm and $\angle BAC = 35°$. Find $\angle ABC$.



Here $a = 5.2$ cm, $c = 3.8$ cm and $A = 35°$. You first need to find angle C.

$\frac{\sin C}{c} = \frac{\sin A}{a}$

$\frac{\sin C}{3.8} = \frac{\sin 35°}{5.2}$

So $\sin C = \frac{3.8 \sin 35°}{5.2}$

$C = 24.781\ldots$

$B = 120°$ (3 s.f.)

Use $\frac{\sin C}{c} = \frac{\sin A}{a}$.
Write the formula you are going to use as the first line of working.

Use your calculator to find the value of C in a single step. Don't round your answer at this point.

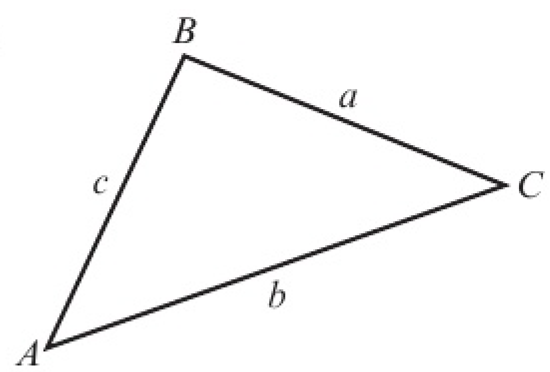
$B = 180° - (24.781\ldots° + 35°) = 120.21\ldots$ which rounds to 120° (3 s.f.)

Exercise 6B

SKILLS ANALYSIS

Give answers to 3 significant figures, where appropriate.

1 In each of parts **a** to **d**, the given values refer to the general triangle.



a Given that $a = 8$ cm, $A = 30°$, $B = 72°$, find b.

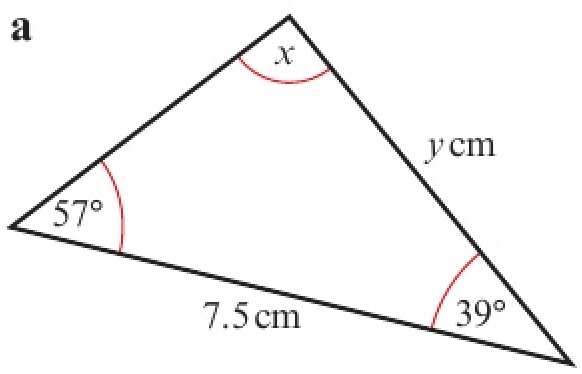
b Given that $a = 24$ cm, $A = 110°$, $C = 22°$, find c.

c Given that $b = 14.7$ cm, $A = 30°$, $C = 95°$, find a.

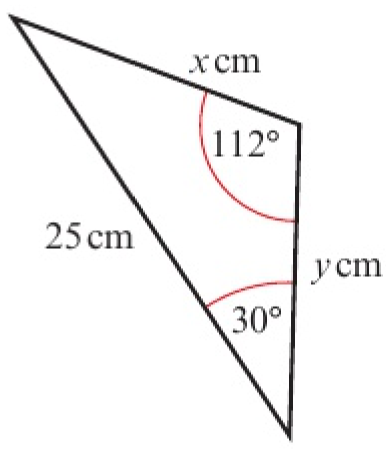
d Given that $c = 9.8$ cm, $B = 68.4°$, $C = 83.7°$, find a.

2 In each of the following triangles, calculate the values of x and y.

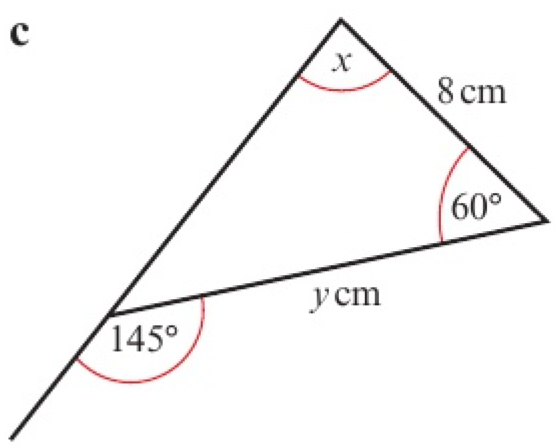
a



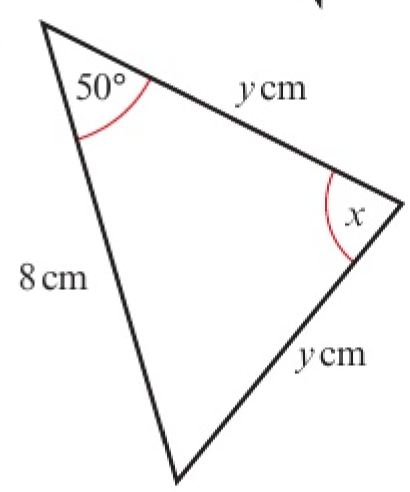
b



c

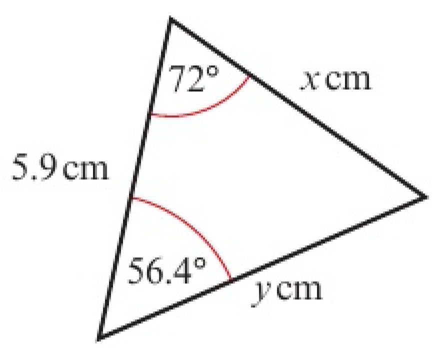


d

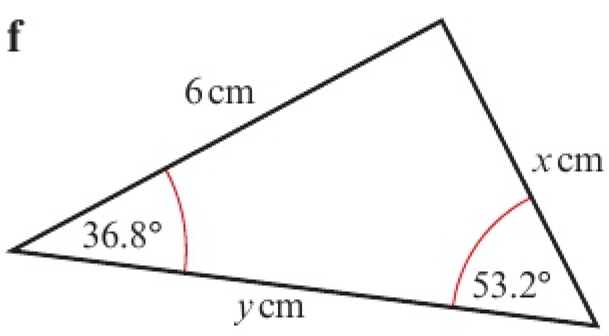


Hint In parts **c** and **d**, start by finding the size of the third angle.

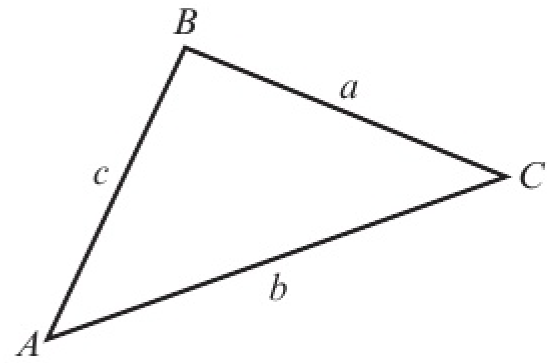
e



f



3 In each of the following sets of data for a triangle ABC, find the value of x.



a $AB = 6$ cm, $BC = 9$ cm, $\angle BAC = 117°$, $\angle ACB = x$

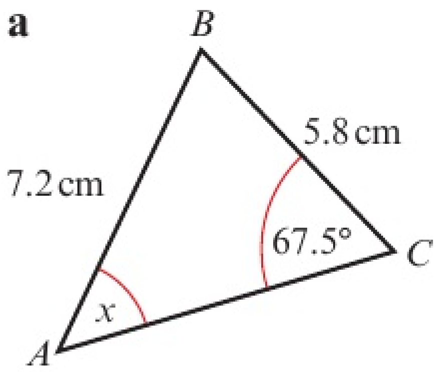
b $AC = 11$ cm, $BC = 10$ cm, $\angle ABC = 40°$, $\angle CAB = x$

c $AB = 6$ cm, $BC = 8$ cm, $\angle BAC = 60°$, $\angle ACB = x$

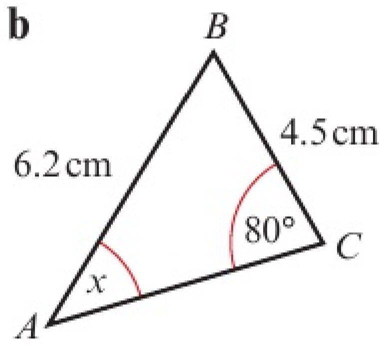
d $AB = 8.7$ cm, $AC = 10.8$ cm, $\angle ABC = 28°$, $\angle BAC = x$

4 In each of the diagrams shown below, work out the size of angle x.

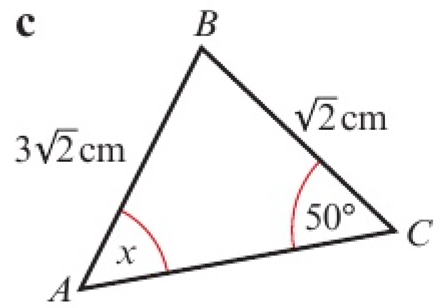
a



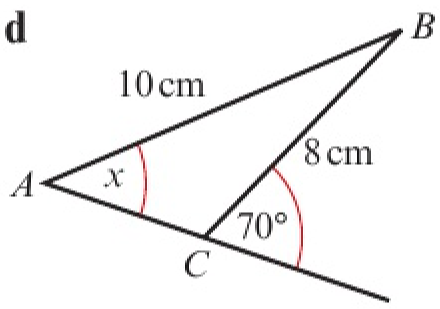
b



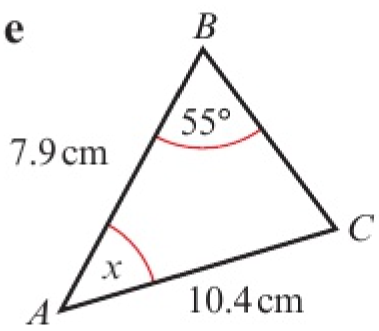
c



d



e



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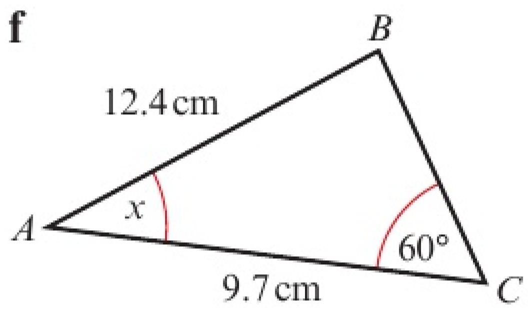
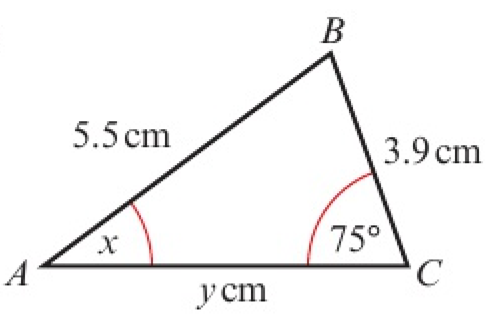


5 In $\triangle PQR$, $QR = \sqrt{3}$ cm, $\angle PQR = 45°$ and $\angle QPR = 60°$. Find **a** PR and **b** PQ.

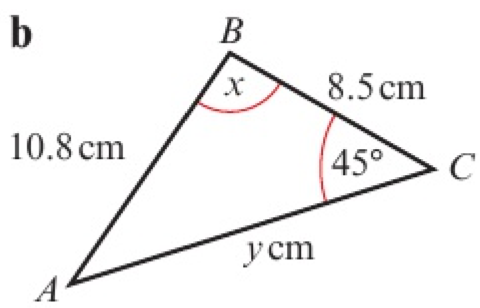
6 In $\triangle PQR$, $PQ = 15$ cm, $QR = 12$ cm and $\angle PRQ = 75°$. Find the two remaining angles.

7 In each of the following diagrams, work out the values of x and y.

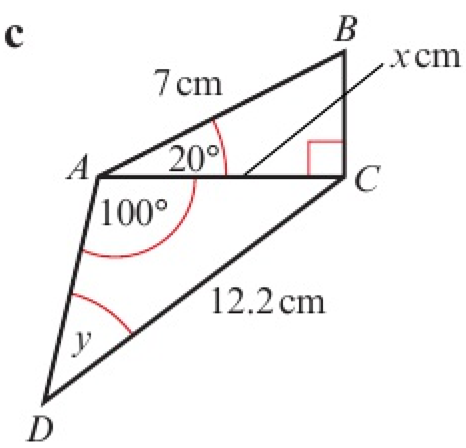
a



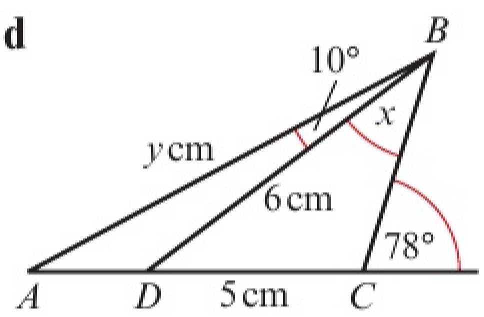
b



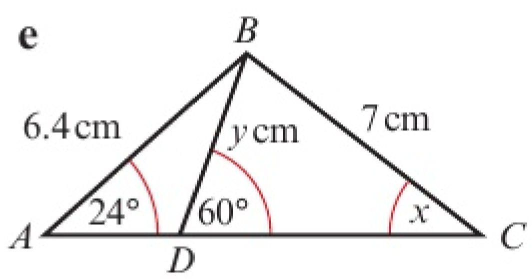
c



d



e



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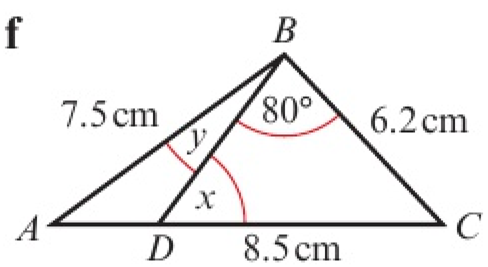


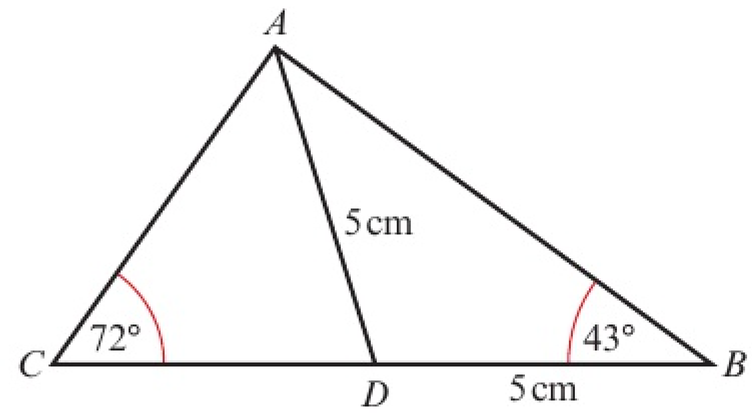
(P) **8** Town B is 6 km, on a bearing of 020°, from town A. Town C is located on a bearing of 055° from town A and on a bearing of 120° from town B. Work out the distance of town C from:

a town A **b** town B

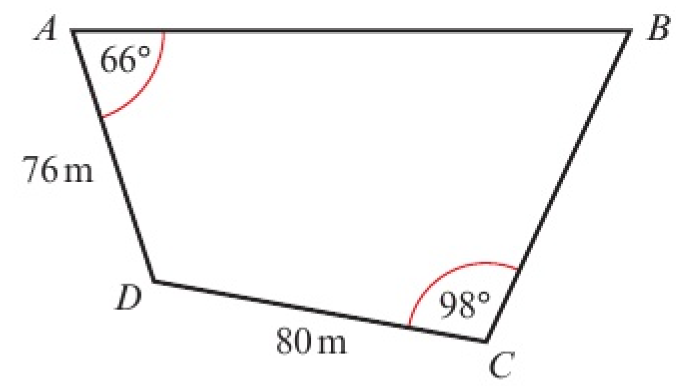
Problem-solving

Draw a sketch to show the information.

9 In the diagram, $AD = DB = 5$ cm, $\angle ABC = 43°$ and $\angle ACB = 72°$.
Calculate:
a AB
b CD



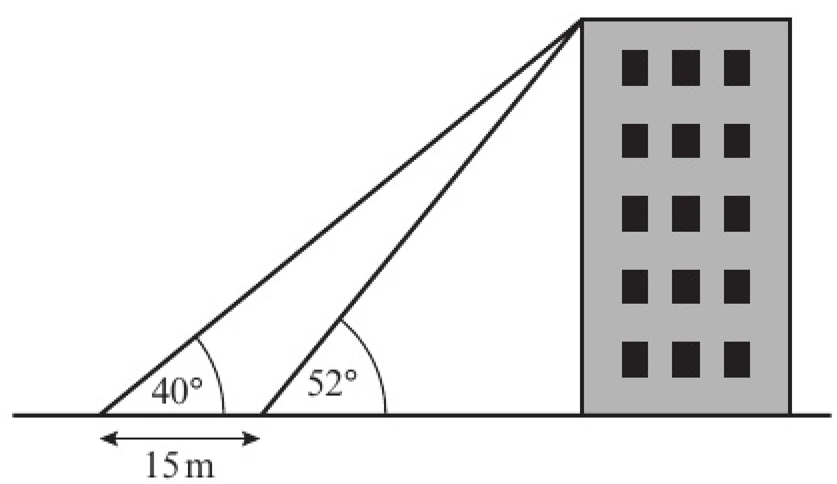
10 A zookeeper is building an enclosure for some llamas. The enclosure is in the shape of a quadrilateral as shown.
If the length of the diagonal BD is 136 m:
a find the angle between the fences AB and BC
b find the length of fence AB.



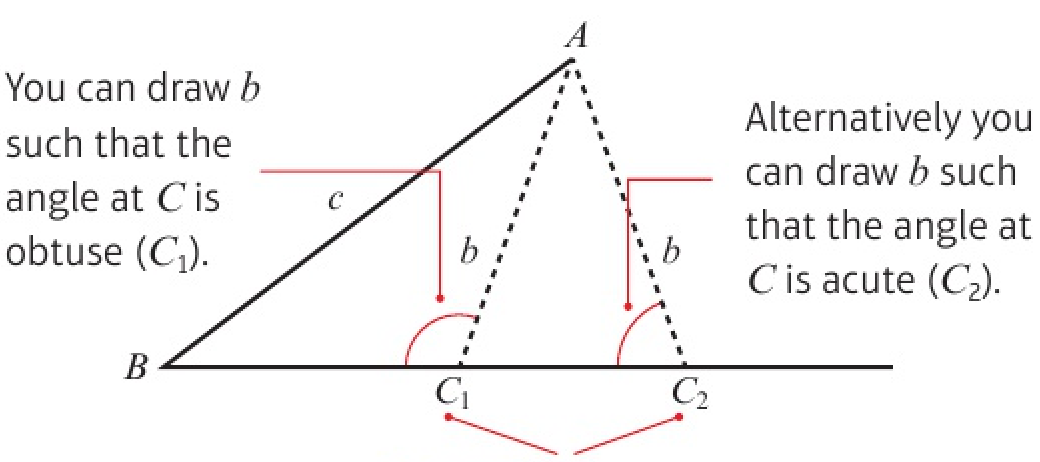
E/P **11** In $\triangle ABC$, $AB = x$ cm, $BC = (4 - x)$ cm, $\angle BAC = y$ and $\angle BCA = 30°$.
Given that $\sin y = \frac{1}{\sqrt{2}}$, show that $x = 4(\sqrt{2} - 1)$. **(5 marks)**

Problem-solving
You can use the value of sin y directly in your calculation. You don't need to work out the value of y.

E/P **12** A surveyor wants to determine the height of a building. She measures the angle of elevation of the top of the building at two points 15 m apart on the ground.
a Use this information to determine the height of the building. **(4 marks)**
b State one assumption made by the surveyor in using this mathematical model. **(1 mark)**



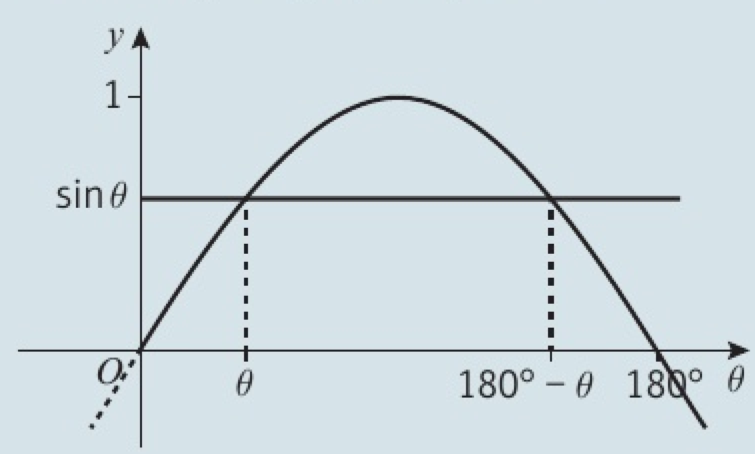
For given side lengths b and c and given angle B, you can draw the triangle in two different ways.



Since AC_1C_2 is an **isosceles** triangle, it follows that the angles AC_1B and AC_2B add together to make 180°.

■ **The sine rule sometimes produces two possible solutions for a missing angle:**
- **$\sin \theta = \sin (180° - \theta)$**

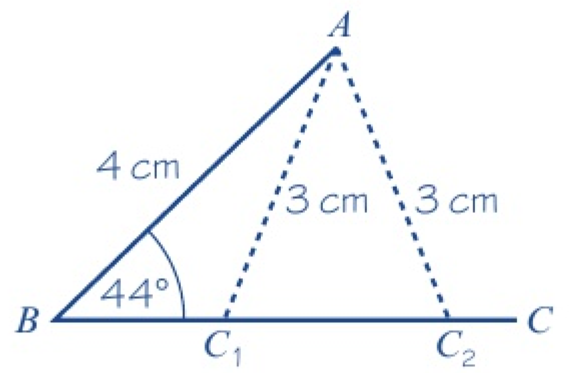
Links You can confirm this relationship by considering the graph of $y = \sin \theta$.



→ Section 6.5

Example 7 SKILLS ANALYSIS

In $\triangle ABC$, $AB = 4$ cm, $AC = 3$ cm and $\angle ABC = 44°$. Work out the two possible values of $\angle ACB$.



Problem-solving

Think about which lengths and angles are fixed, and which ones can vary. The length AC is fixed. If you drew a circle with **radius** 3 cm and centre A it would intersect the horizontal side of the triangle at two points, C_1 and C_2.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

Use $\frac{\sin C}{c} = \frac{\sin B}{b}$, where $b = 3$, $c = 4$, $B = 44°$.

$$\frac{\sin C}{4} = \frac{\sin 44°}{3}$$

$$\sin C = \frac{4 \sin 44°}{3}$$

So $C = 67.851... = 67.9°$ (3 s.f.)

or $C = 180 - 67.851... = 112.14...$
$= 112°$ (3 s.f.)

As $\sin(180° - \theta) = \sin\theta$.

Exercise 6C SKILLS ANALYSIS

Give answers to 3 significant figures, where appropriate.

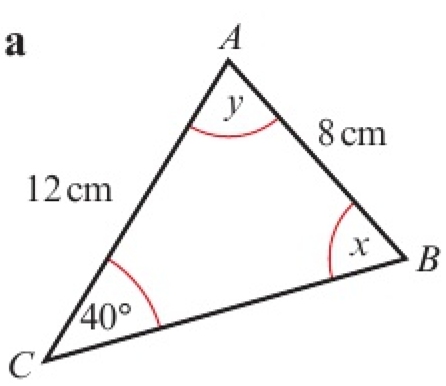
1 In $\triangle ABC$, $BC = 6$ cm, $AC = 4.5$ cm and $\angle ABC = 45°$.

 a Calculate the two possible values of $\angle BAC$.

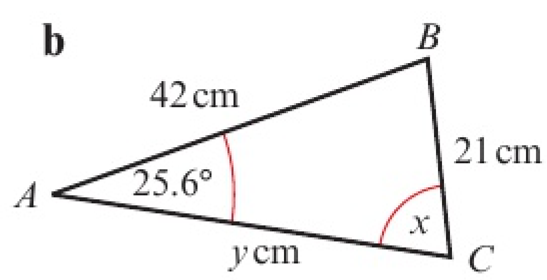
 b Draw a diagram to illustrate your answers.

2 In each of the diagrams below, calculate the possible values of x and the corresponding values of y.

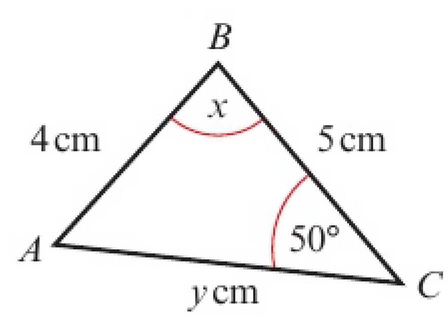
a



b



c



(P) 3 In each of the following cases, $\triangle ABC$ has $\angle ABC = 30°$ and $AB = 10$ cm.

 a Calculate the least possible length that AC could be.

 b Given that $AC = 12$ cm, calculate $\angle ACB$.

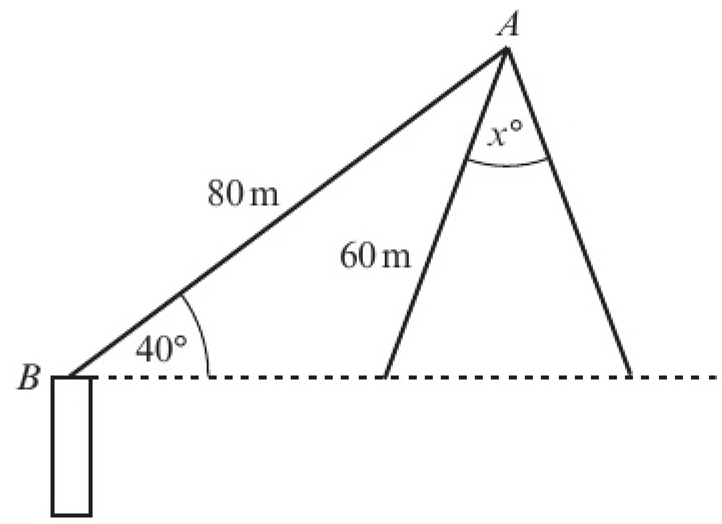
 c Given instead that $AC = 7$ cm, calculate the two possible values of $\angle ACB$.

(P) **4** Triangle ABC is such that $AB = 4$ cm, $BC = 6$ cm and $\angle ACB = 36°$. Show that one of the possible values of $\angle ABC$ is 25.8° (to 3 s.f.). Using this value, calculate the length of AC.

(P) **5** Two triangles ABC are such that $AB = 4.5$ cm, $BC = 6.8$ cm and $\angle ACB = 30°$. Work out the value of the largest angle in each of the triangles.

(E/P) **6** **a** A crane arm AB of length 80 m is anchored at point B at an angle of 40° to the horizontal. A wrecking ball is suspended on a cable of length 60 m from A. Find the angle x through which the wrecking ball rotates as it passes the two points level with the base of the crane arm at B. **(6 marks)**

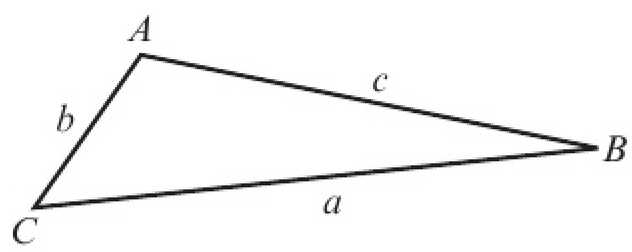
b Write down one modelling assumption you have made. **(1 mark)**



6.3 Areas of triangles

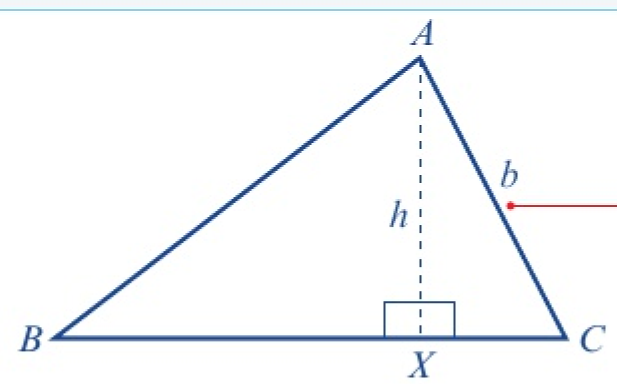
You need to be able to use the formula for finding the area of any triangle when you know two sides and the angle between them.

- Area $= \frac{1}{2}ab \sin C$



Hint As with the cosine rule, the letters are interchangeable. For example, if you know angle B and sides a and c, the formula becomes Area $= \frac{1}{2}ac \sin B$.

A proof of the formula:



The perpendicular from A to BC is drawn and it meets BC at X. The length of $AX = h$.

Area of $\triangle ABC = \frac{1}{2}ah$ (1)

But $h = b \sin C$ (2)

So Area $= \frac{1}{2}ab \sin C$

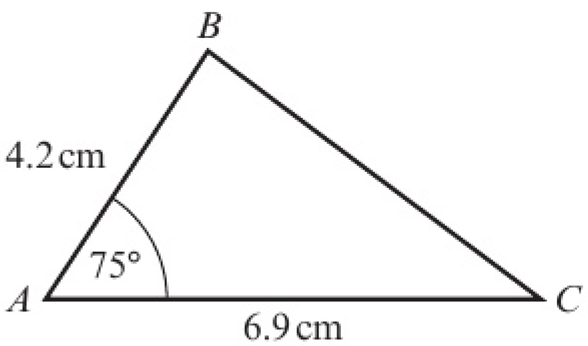
Area of triangle $= \frac{1}{2}$ base × height.

Use the sine ratio $\sin C = \frac{h}{b}$ in $\triangle AXC$.

Substitute (2) into (1).

Example 8

Work out the area of the triangle shown below.



Online Explore the area of a triangle using technology.

Area $= \frac{1}{2}bc \sin A$

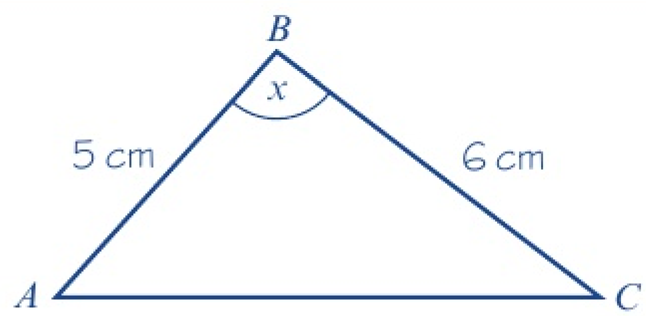
Area of $\triangle ABC = \frac{1}{2} \times 6.9 \times 4.2 \times \sin 75°\,\text{cm}^2$

$= 14.0\,\text{cm}^2$ (3 s.f.)

Here $b = 6.9$ cm, $c = 4.2$ cm and angle $A = 75°$, so use: Area $= \frac{1}{2}bc \sin A$.

Example 9

In $\triangle ABC$, $AB = 5$ cm, $BC = 6$ cm and $\angle ABC = x$. Given that the area of $\triangle ABC$ is $12\,\text{cm}^2$ and that AC is the longest side, find the value of x.



Area $= \frac{1}{2}ac \sin B$

Area of $\triangle ABC = \frac{1}{2} \times 6 \times 5 \times \sin x\,\text{cm}^2$

So $12\,\text{cm}^2 = \frac{1}{2} \times 6 \times 5 \times \sin x\,\text{cm}^2$

$\sin x = 0.8$

$x = 126.86\ldots$

$= 127°$ (3 s.f.)

Here $a = 6$ cm, $c = 5$ cm and angle $B = x$, so use: Area $= \frac{1}{2}ac \sin B$.

Area of $\triangle ABC$ is $12\,\text{cm}^2$.

$\sin x = \frac{12}{15}$

Problem-solving

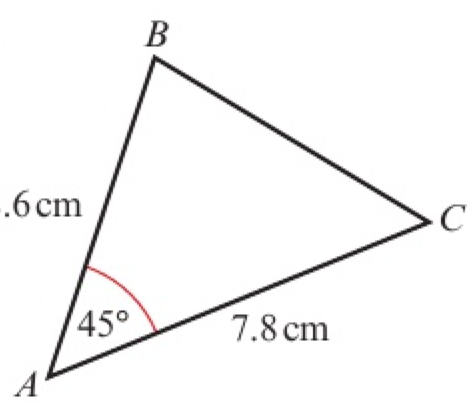
There are two values of x for which $\sin x = 0.8$: $53.13\ldots°$ and $126.86\ldots°$. But here you know B is the largest angle because AC is the largest side.

Exercise 6D

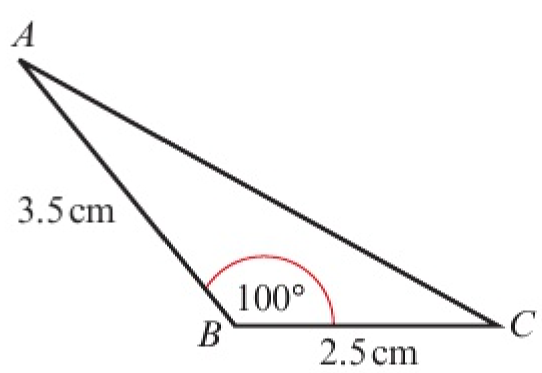
SKILLS ANALYSIS

1 Calculate the area of each triangle.

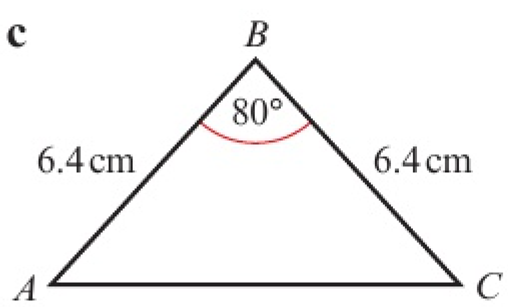
a



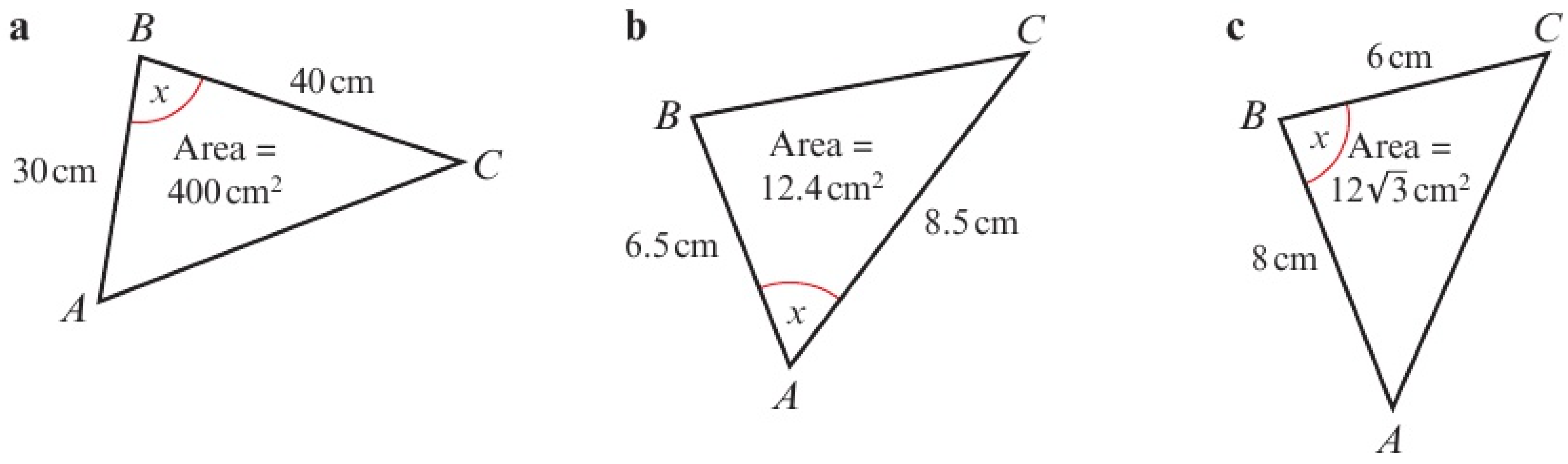
b



c

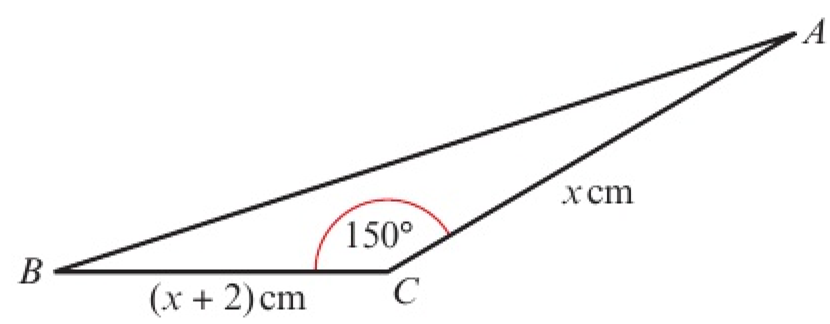


2 Work out the possible sizes of x in the following triangles.



3 A fenced triangular plot of ground has area 1200 m^2. The fences along the two smaller sides are 60 m and 80 m respectively and the angle between them is θ. Show that $\theta = 150°$, and work out the total length of fencing.

(P) **4** In triangle ABC, $BC = (x + 2)$ cm, $AC = x$ cm and $\angle BCA = 150°$. Given that the area of the triangle is 5 cm^2, work out the value of x, giving your answer to 3 significant figures.



(E/P) **5** In $\triangle PQR$, $PQ = (x + 2)$ cm, $PR = (5 - x)$ cm and $\angle QPR = 30°$. The area of the triangle is A cm^2.

a Show that $A = \frac{1}{4}(10 + 3x - x^2)$. **(3 marks)**

b Use the method of completing the square, or otherwise, to find the maximum value of A, and give the corresponding value of x. **(4 marks)**

(E/P) **6** In $\triangle ABC$, $AB = x$ cm, $AC = (5 + x)$ cm and $\angle BAC = 150°$. Given that the area of the triangle is $3\frac{3}{4}$ cm^2,

a show that x satisfies the equation $x^2 + 5x - 15 = 0$ **(3 marks)**

b calculate the value of x, giving your answer to 3 significant figures. **(3 marks)**

Problem-solving
x represents a length so it must be positive.

6.4 Solving triangle problems

You can solve problems involving triangles by using the sine and cosine rules along with Pythagoras' theorem and standard right-angled triangle trigonometry.

If some of the triangles are right-angled, try to use basic trigonometry and Pythagoras' theorem first to work out other information.

If you encounter a triangle which is not right-angled, you will need to decide whether to use the sine rule or the cosine rule. Generally, use the sine rule when you are considering two angles and two sides and the cosine rule when you are considering three sides and one angle.

Watch out The sine rule is often easier to use than the cosine rule. If you know one side and an opposite angle in a triangle, try to use the sine rule to find other missing sides and angles.

For questions involving area, check first if you can use Area = $\frac{1}{2}$ × base × height, before using the formula involving sine.

- To find an unknown angle given two sides and one opposite angle, use the sine rule.
- To find an unknown side given two angles and one opposite side, use the sine rule.
- To find an unknown angle given all three sides, use the cosine rule.
- To find an unknown side given two sides and the angle between them, use the cosine rule.
- To find the area given two sides and the angle between them, use Area = $\frac{1}{2}ab \sin C$.

Example 10 SKILLS PROBLEM SOLVING

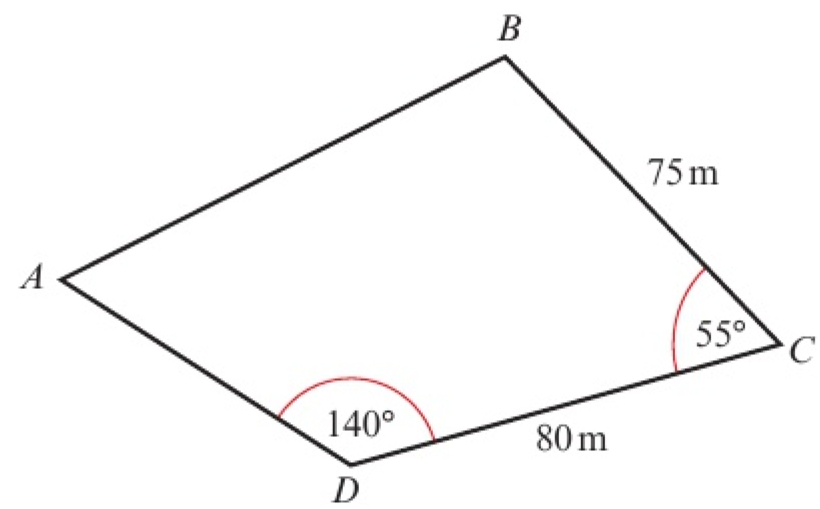
The diagram shows the locations of four mobile phone masts in a field. $BC = 75$ m, $CD = 80$ m, angle $BCD = 55°$ and angle $ADC = 140°$.

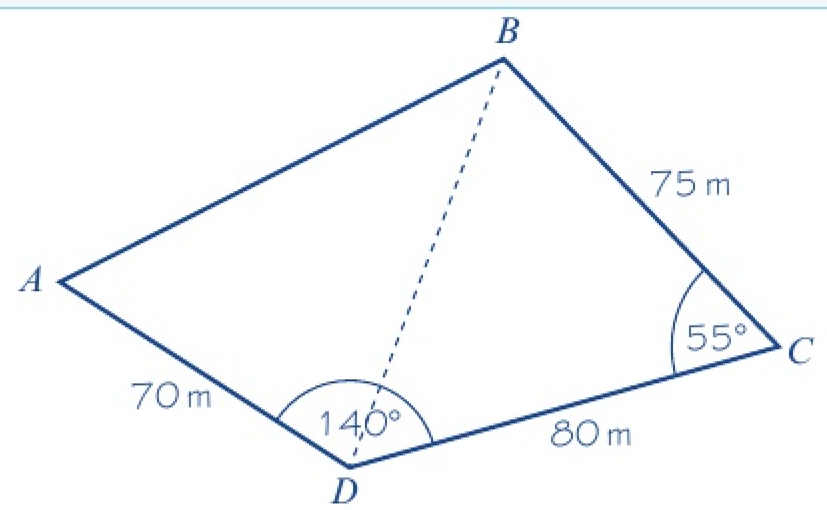
In order that the masts do not interfere with each other, they must be at least 70 m apart.
Given that A is the minimum distance from D, find:

a the distance from A to B

b the angle BAD

c the area enclosed by the four masts.





Problem-solving

Split the diagram into two triangles. Use the information in triangle BCD to work out the length BD. You are using three sides and one angle so use the **cosine rule**.

a $BD^2 = BC^2 + CD^2 - 2(BC)(CD)\cos \angle BCD$

Find BD first using the cosine rule.

$BD^2 = 75^2 + 80^2 - 2(75)(80)\cos 55°$

$BD^2 = 5142.08\ldots$

$BD = \sqrt{5142.08\ldots} = 71.708\ldots$

Store this value in your calculator, or write down all the digits from your calculator display.

$$\frac{\sin \angle BDC}{BC} = \frac{\sin \angle BCD}{BD}$$

You know a side and its opposite angle (BD and $\angle BCD$), so use the sine rule to calculate $\angle BDC$.

$$\sin \angle BDC = \frac{\sin(55°) \times 75}{71.708} = 0.85675\ldots$$

$\angle BDC = 58.954\ldots$

$\angle BDA = 140 - 58.954\ldots = 81.045\ldots$

Find BDA and store this value, or write down all the digits from your calculator display.

$AB^2 = AD^2 + BD^2 - 2(AD)(BD)\cos\angle BDA$

$AB^2 = 70^2 + 71.708\ldots^2 - 2(70)(71.708\ldots)\cos(81.045\ldots)$

$AB^2 = 8479.55\ldots$

$AB = \sqrt{8479.55\ldots} = 92.084\ldots$

$= 92.1$ m (3 s.f.)

You can now use the cosine rule in triangle ABD to find AB.

AB is a length, so you are only interested in the positive solution.

b $\dfrac{\sin\angle BAD}{BD} = \dfrac{\sin\angle BDA}{AB}$

$\sin\angle BAD = \dfrac{\sin(81.045\ldots) \times 71.708\ldots}{92.084\ldots}$

$= 0.769\ldots$

$\angle BAD = 50.28\ldots = 50.3°$ (3 s.f.)

Use the sine rule to calculate angle BAD.

Alternatively you could have used the cosine rule with sides AB, BD and AD.

c Area $ABCD$ = area BCD + area BDA

Area $ABCD = \frac{1}{2}(BC)(CD)\sin\angle BCD + \frac{1}{2}(AB)(AD)\sin\angle BAD$

Area $ABCD = \frac{1}{2}(75)(80)\sin(55°) + \frac{1}{2}(92.084\ldots)(70)\sin(50.28\ldots°)$

Area $ABCD = 2457.4\ldots + 2479.2\ldots$

Area $ABCD = 4936.6\ldots = 4940$ m^2 (3 s.f.)

Use the area formula twice.

Online Explore the solution step-by-step using technology.

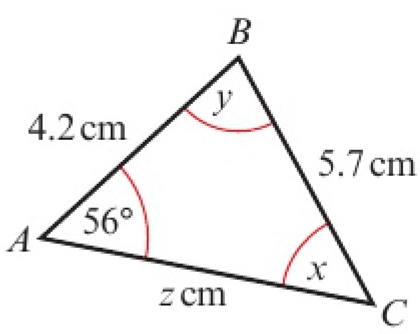
Exercise 6E

SKILLS PROBLEM SOLVING

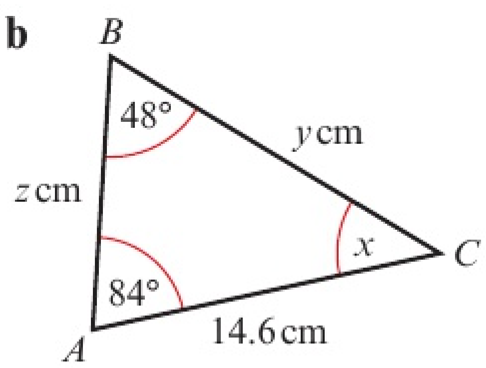
Try to use the most efficient method, and give answers to 3 significant figures.

1 In each triangle below find the values of x, y and z.

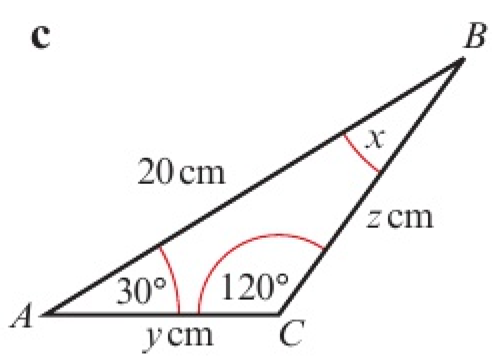
a



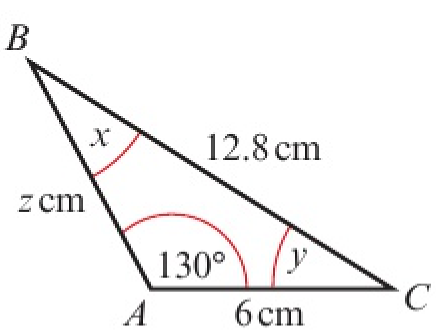
b



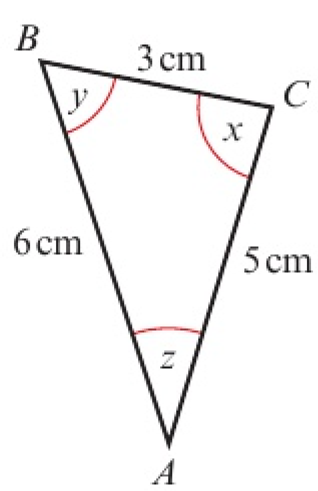
c



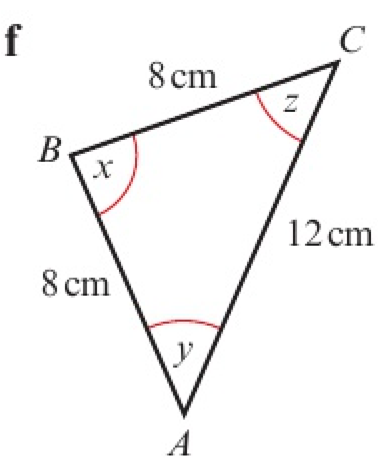
d



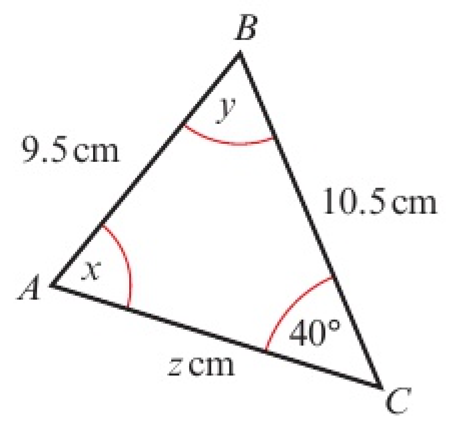
e



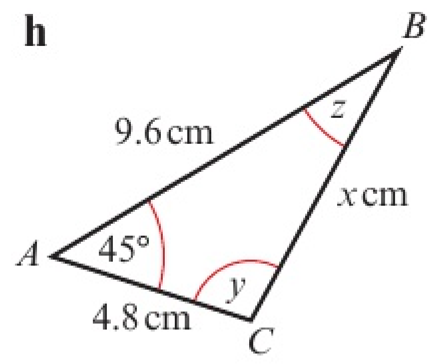
f



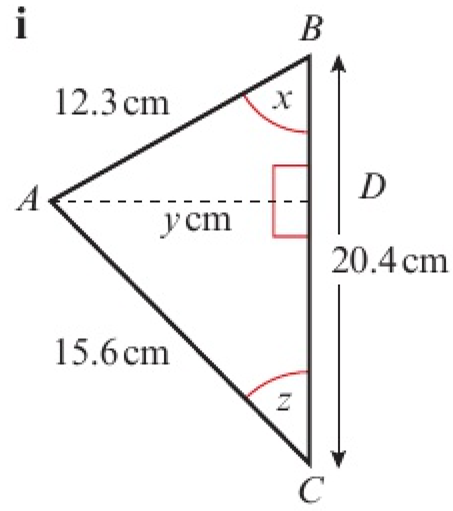
g



h



i



2 In $\triangle ABC$, calculate the size of the remaining angles, the lengths of the third side and the area of the triangle given that

a $\triangle BAC = 40°$, $AB = 8.5$ cm and $BC = 10.2$ cm

b $\triangle ACB = 110°$, $AC = 4.9$ cm and $BC = 6.8$ cm

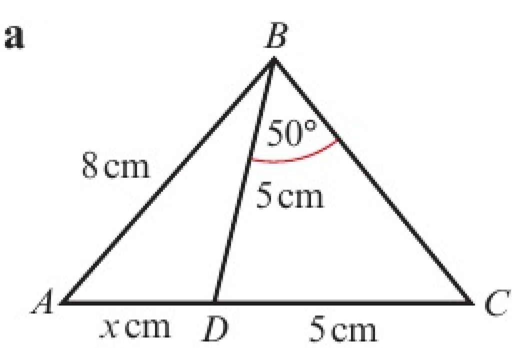
3 A hiker walks due north from A and after 8 km reaches B. She then walks a further 8 km on a bearing of 120° to C. Work out **a** the distance from A to C and **b** the bearing of C from A.

(P) **4** A helicopter flies on a bearing of 200° from A to B, where $AB = 70$ km. It then flies on a bearing of 150° from B to C, where C is due south of A. Work out the distance of C from A.

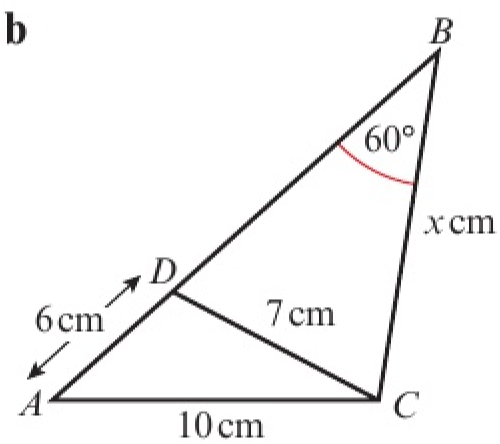
(P) **5** Two radar stations A and B are 16 km apart and A is due north of B. A ship is known to be on a bearing of 150° from A and 10 km from B. Show that this information gives two positions for the ship, and calculate the distance between these two positions.

(P) **6** Find x in each of the following diagrams:

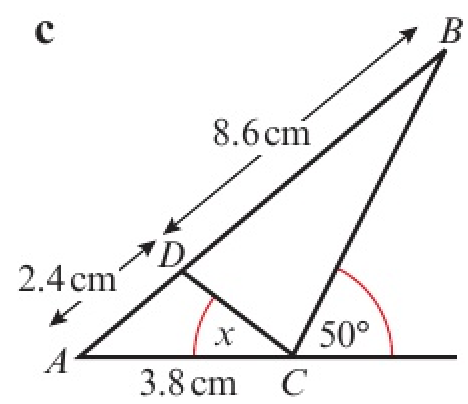
a



b



c



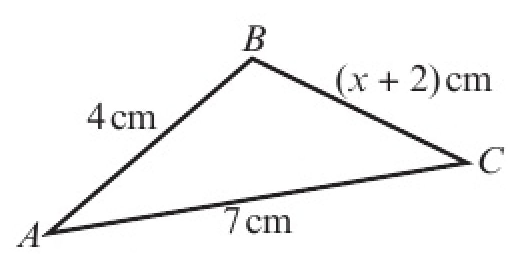
(P) **7** In $\triangle ABC$, $AB = 4$ cm, $BC = (x + 2)$ cm and $AC = 7$ cm.

a Explain how you know that $1 < x < 9$.

b Work out the value of x and the area of the triangle for the cases when

i $\angle ABC = 60°$ and

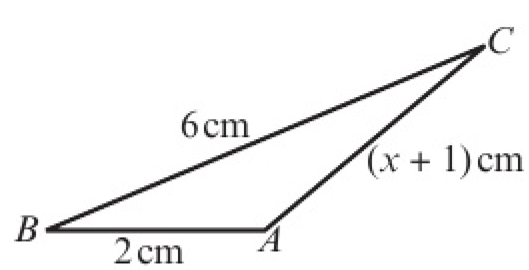
ii $\angle ABC = 45°$, giving your answers to 3 significant figures.



(P) **8** In the triangle, $\cos \angle ABC = \frac{5}{8}$.

a Calculate the value of x.

b Find the area of triangle ABC.



(P) **9** In $\triangle ABC$, $AB = \sqrt{2}$ cm, $BC = \sqrt{3}$ cm and $\angle BAC = 60°$. Show that $\angle ACB = 45°$ and find AC.

(P) **10** In $\triangle ABC$, $AB = (2 - x)$ cm, $BC = (x + 1)$ cm and $\angle ABC = 120°$.

a Show that $AC^2 = x^2 - x + 7$.

b Find the value of x for which AC has a minimum value.

Problem-solving

Complete the square for the expression $x^2 - x + 7$ to find the minimum value of AC^2 and the value of x where it occurs.

11 Triangle ABC is such that $BC = 5\sqrt{2}$ cm, $\angle ABC = 30°$ and $\angle BAC = \theta$, where $\sin\theta = \frac{\sqrt{5}}{8}$.

Work out the length of AC, giving your answer in the form $a\sqrt{b}$, where a and b are integers.

(P) **12** The **perimeter** of $\triangle ABC = 15$ cm. Given that $AB = 7$ cm and $\angle BAC = 60°$, find the lengths of AC and BC and the area of the triangle.

(E) **13** In the triangle ABC, $AB = 14$ cm, $BC = 12$ cm and $CA = 15$ cm.

a Find the size of angle C, giving your answer to 3 s.f. **(3 marks)**

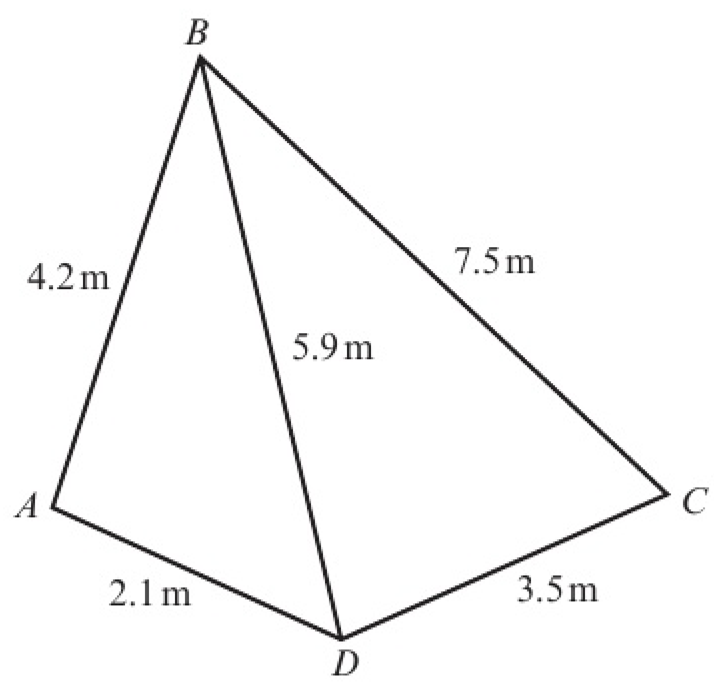
b Find the area of triangle ABC, giving your answer in cm^2 to 3 s.f. **(3 marks)**

(E/P) **14** A flower bed is in the shape of a quadrilateral as shown in the diagram.

a Find the sizes of angles DAB and BCD. **(4 marks)**

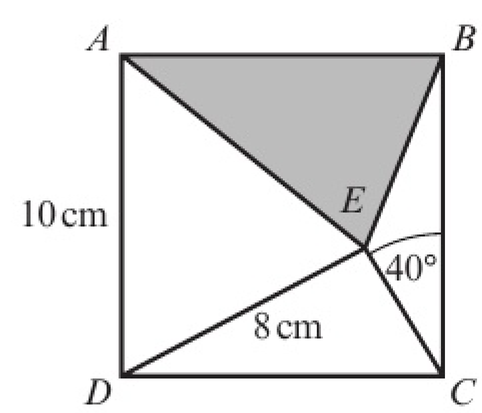
b Find the total area of the flower bed. **(3 marks)**

c Find the length of the diagonal AC. **(4 marks)**



(E/P) **15** $ABCD$ is a square. Angle CED is obtuse.

Find the area of the shaded triangle. **(7 marks)**

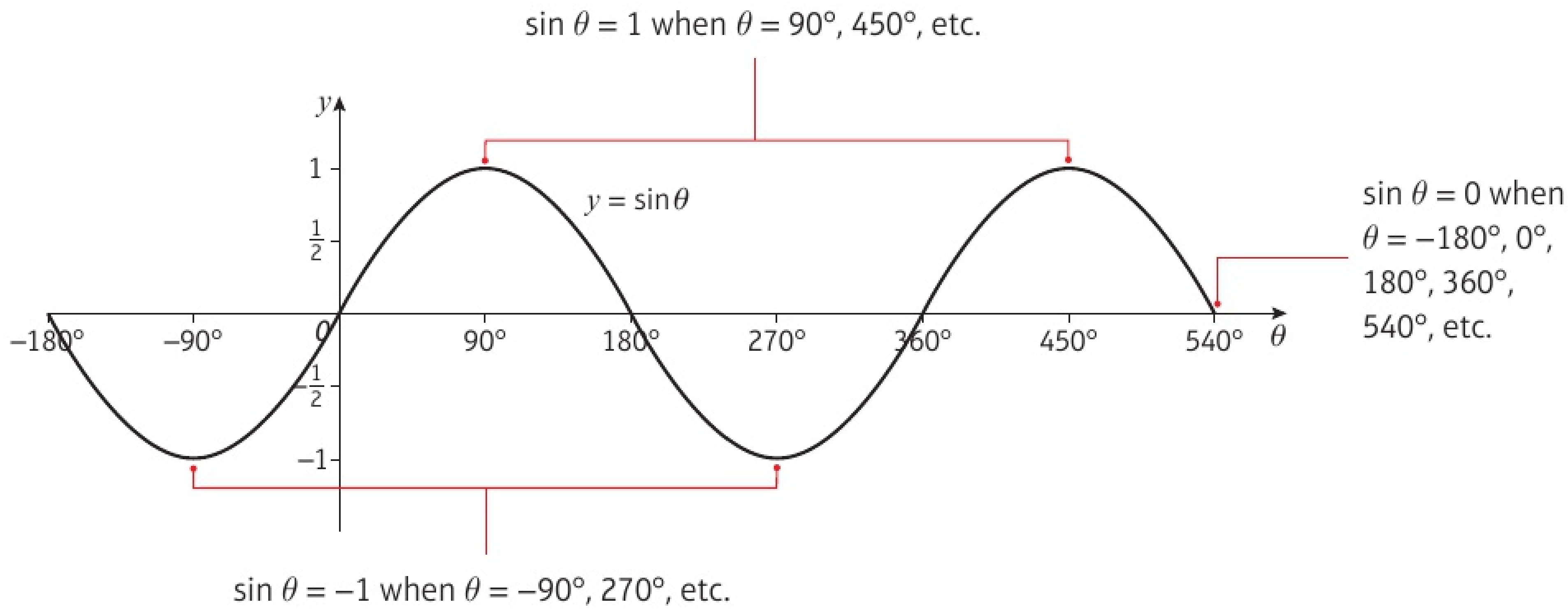


6.5 Graphs of sine, cosine and tangent

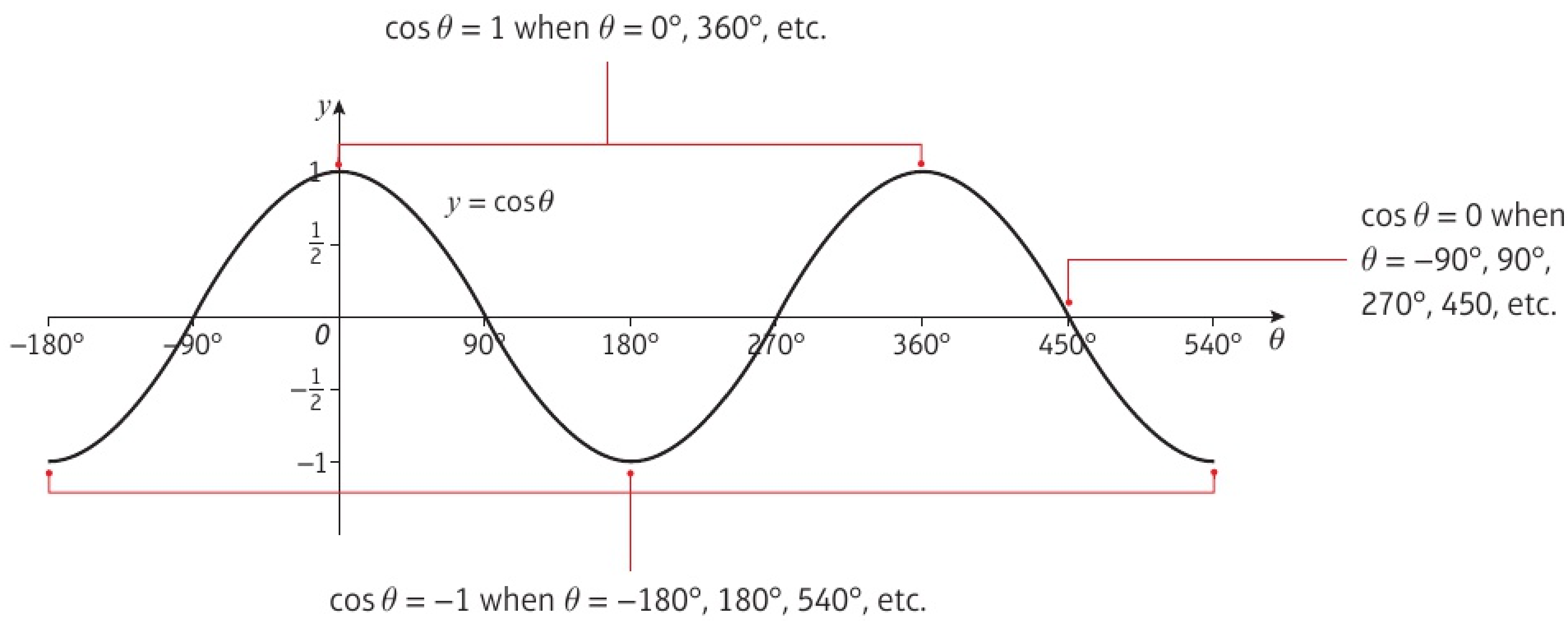
- The graphs of sine, cosine and tangent are periodic.
  They repeat themselves after a certain interval **period**.

You need to be able to draw the graphs for a given range of angles.

- The graph of $y = \sin\theta$:
  - repeats itself every 360° period and crosses the x-axis at …, –180°, 0°, 180°, 360°, …
  - has a maximum value of 1 and a minimum value of –1.

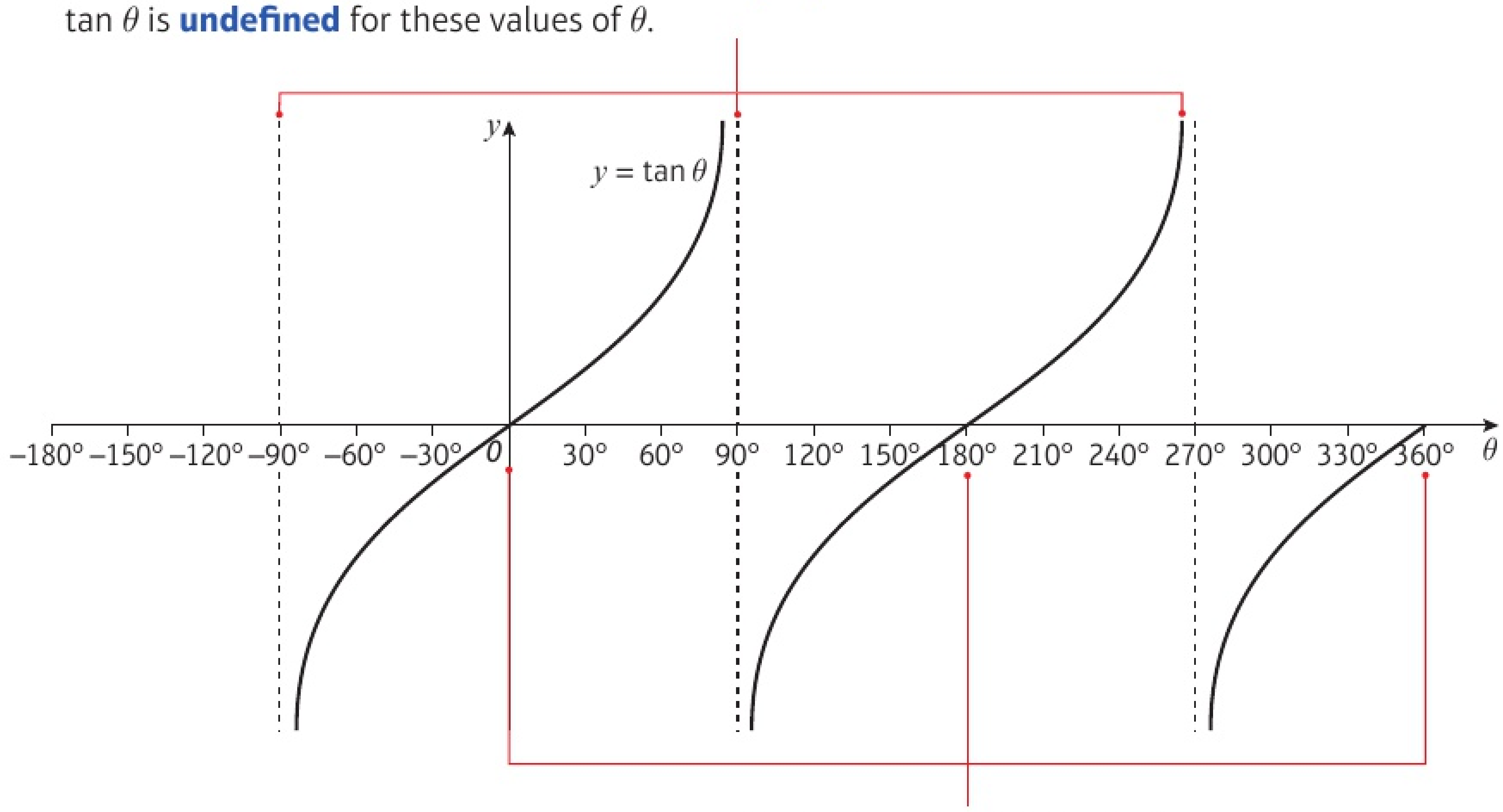


- The graph of $y = \cos\theta$:
  - repeats itself every 360° period and crosses the x-axis at …, –90°, 90°, 270°, 450°, …
  - has a maximum value of 1 and a minimum value of –1.



- The graph of $y = \tan \theta$:
  - repeats itself every 180° period and crosses the x-axis at … –180°, 0°, 180°, 360°, …
  - has no maximum or minimum value
  - has vertical asymptotes at $x = -90°$, $x = 90°$, $x = 270°$, …

tan θ does *not* have maximum and minimum points but approaches negative or positive infinity as the curve approaches the **asymptotes** at –90°, 90°, 270°, etc. tan θ is **undefined** for these values of θ.



tan θ = 0 when θ = 0°, 180°, 360°, etc.

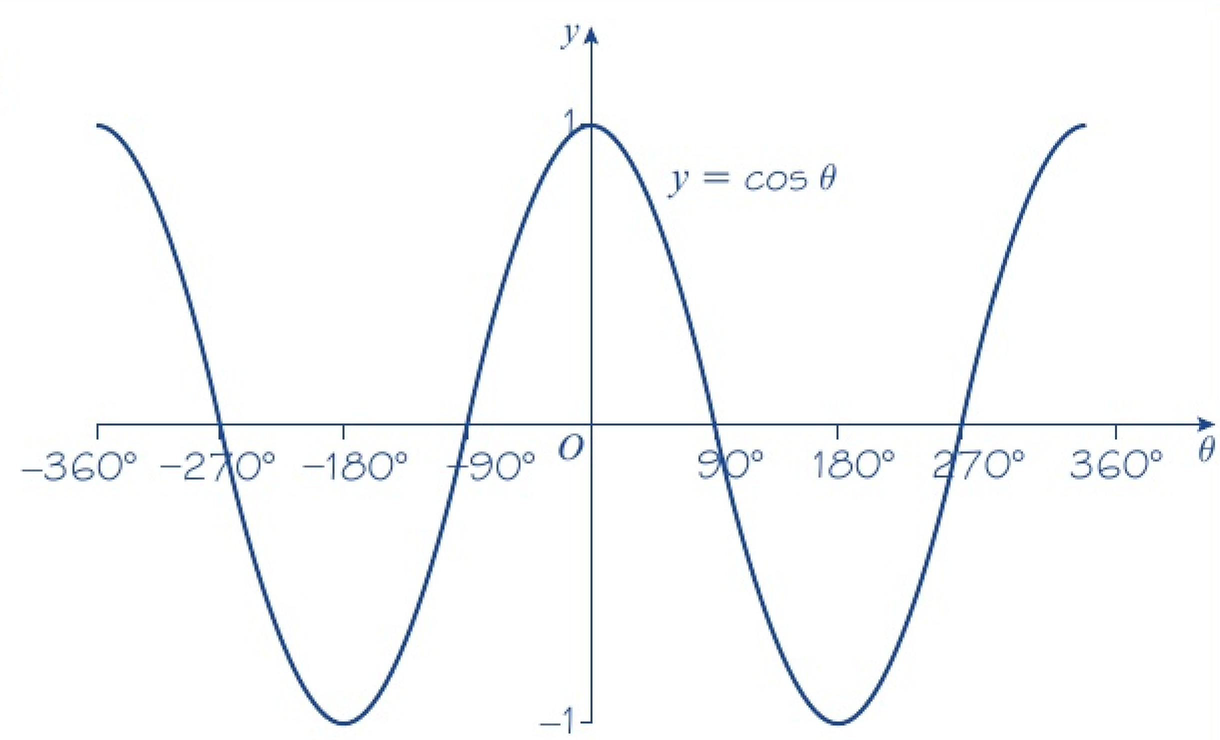
Example 11

a Sketch the graph of $y = \cos \theta$ in the interval $-360° \leqslant \theta \leqslant 360°$.

b **i** Sketch the graph of $y = \sin x$ in the interval $-180° \leqslant x \leqslant 270°$.

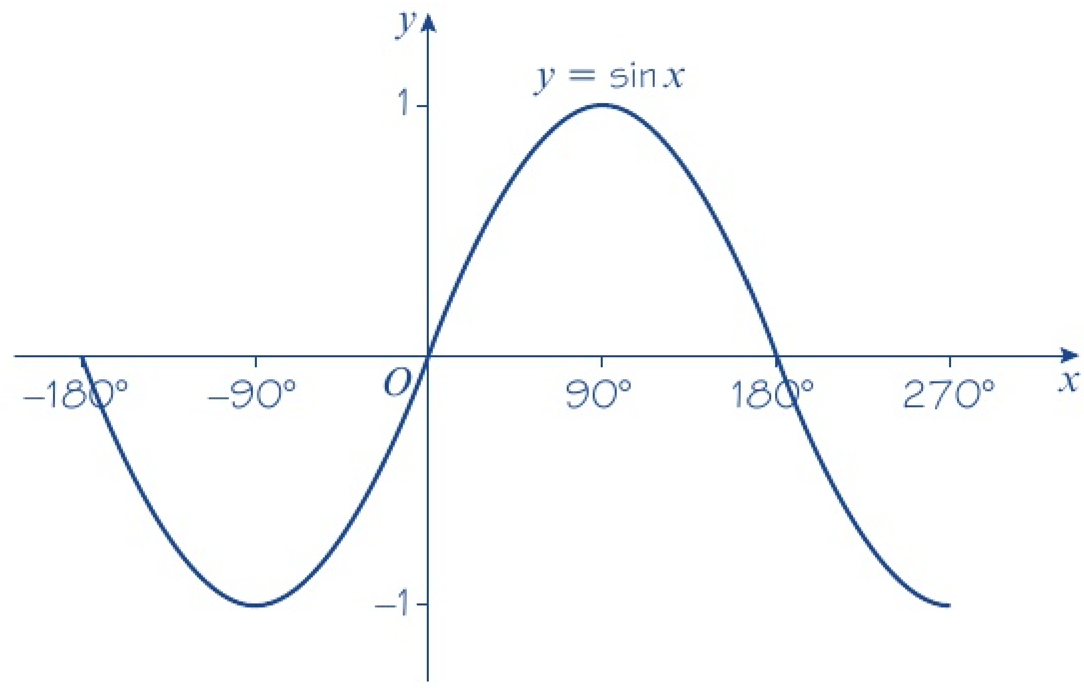
ii $\sin(-30°) = -0.5$. Use your graph to determine two further values of x for which $\sin x = -0.5$.

a



The axes are θ and y.
The curve meets the θ-axis at $\theta = \pm 270°$ and $\theta = \pm 90°$.
The curve crosses the y-axis at (0, 1).

b i



ii Using the symmetry of the graph:

sin (−150°) = −0.5

sin 210° = −0.5

x = −150° or 210°

The line $x = -90°$ is a line of symmetry.

You could also find this value by working out sin (180° − (−30°)).

Exercise 6F

SKILLS INTERPRETATION

1 Sketch the graph of $y = \cos\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

2 Sketch the graph of $y = \tan\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

3 Sketch the graph of $y = \sin\theta$ in the interval $-180° \leqslant \theta \leqslant 180°$.

4 **a** $\cos 30° = \frac{\sqrt{3}}{2}$. Use your graph in question **1** to find another value of θ for which $\cos\theta = \frac{\sqrt{3}}{2}$.

b $\tan 60° = \sqrt{3}$. Use your graph in question **2** to find other values of θ for which:

i $\tan\theta = \sqrt{3}$ **ii** $\tan\theta = -\sqrt{3}$

c $\sin 45° = \frac{1}{\sqrt{2}}$. Use your graph in question **3** to find other values of θ for which:

i $\sin\theta = \frac{1}{\sqrt{2}}$ **ii** $\sin\theta = -\frac{1}{\sqrt{2}}$

6.6 Transforming trigonometric graphs

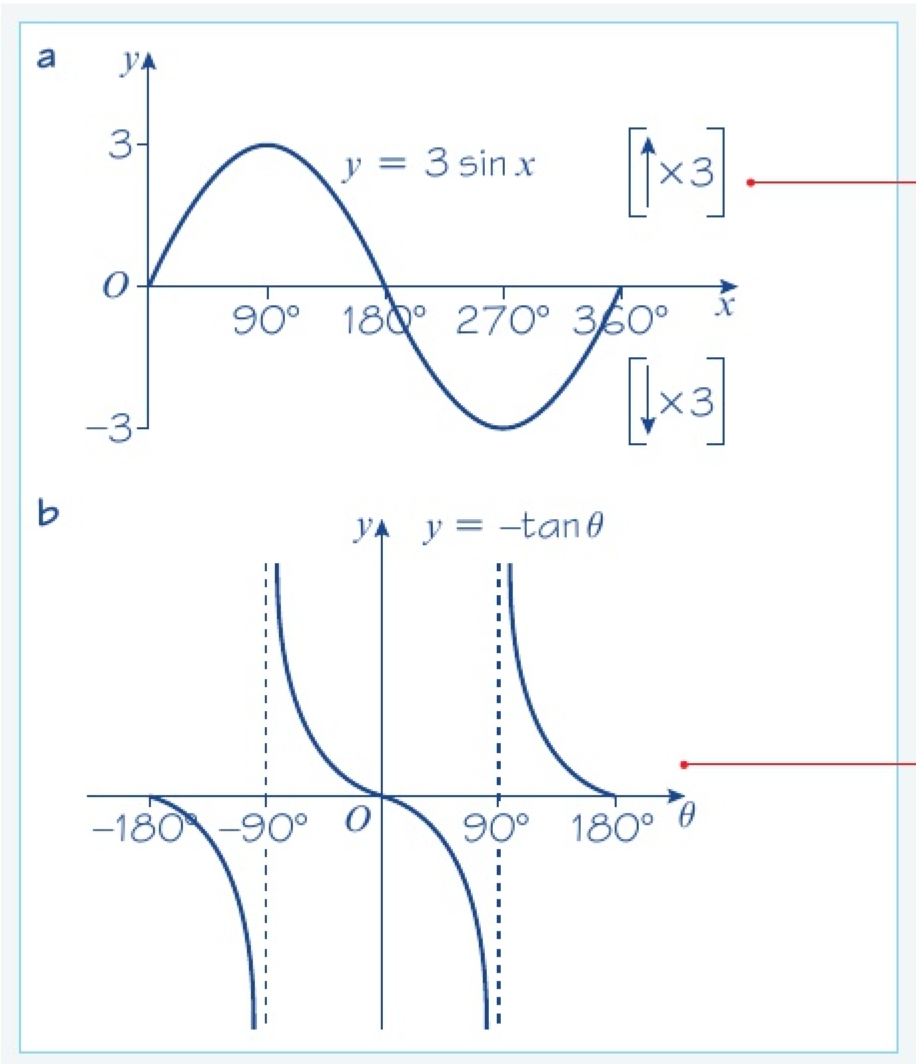
You can use your knowledge of transforming graphs to transform the graphs of trigonometric functions.

Links You need to be able to apply translations and stretches to graphs of trigonometric functions. ← Chapter 4

Example 12

Sketch on separate sets of axes the graphs of:

a $y = 3\sin x, 0 \leqslant x \leqslant 360°$ **b** $y = -\tan\theta, -180° \leqslant \theta \leqslant 180°$



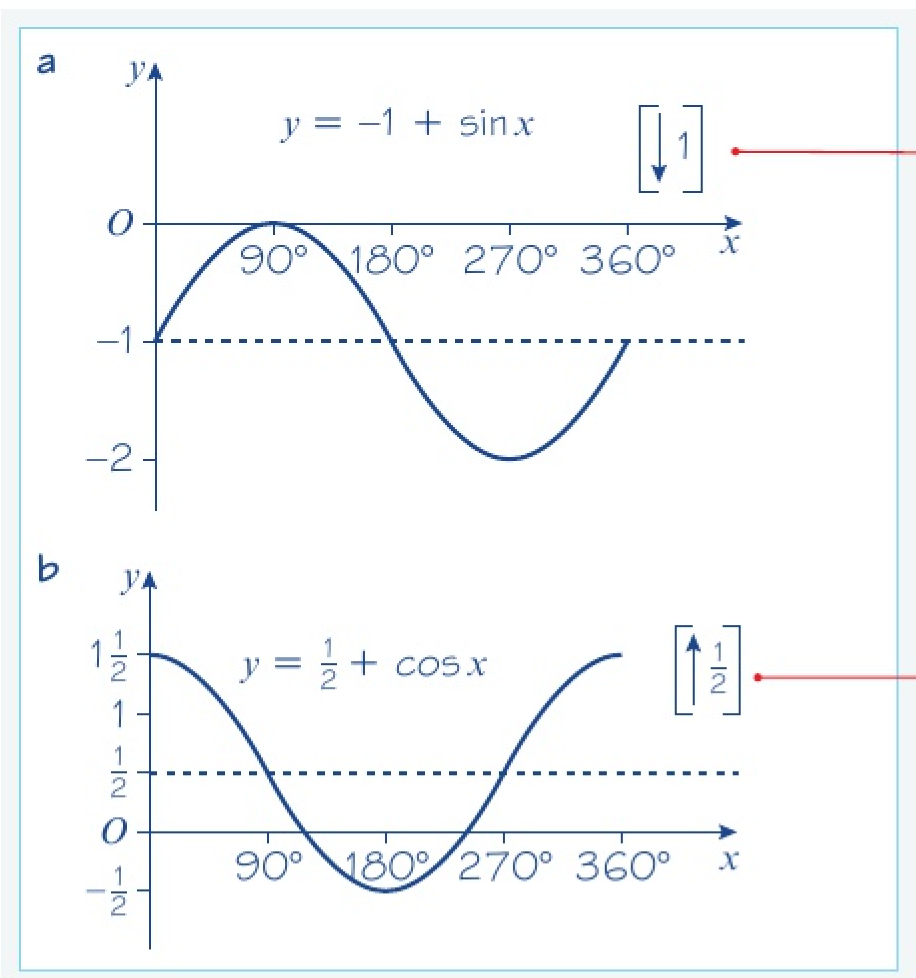
$y = 3f(x)$ is a vertical stretch of the graph $y = f(x)$ with scale factor 3. The intercepts on the x-axis remain unchanged, and the graph has a maximum point at (90°, 3) and a minimum point at (270°, −3).

$y = -f(x)$ is a reflection of the graph $y = f(x)$ in the x-axis. So this graph is a reflection of the graph $y = \tan x$ in the x-axis.

Example 13

Sketch on separate sets of axes the graphs of:

a $y = -1 + \sin x, 0 \leqslant x \leqslant 360°$ **b** $y = \frac{1}{2} + \cos x, 0 \leqslant x \leqslant 360°$



$y = f(x) - 1$ is a translation of the graph $y = f(x)$ by vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

The graph of $y = \sin x$ is translated by 1 unit in the negative y-direction.

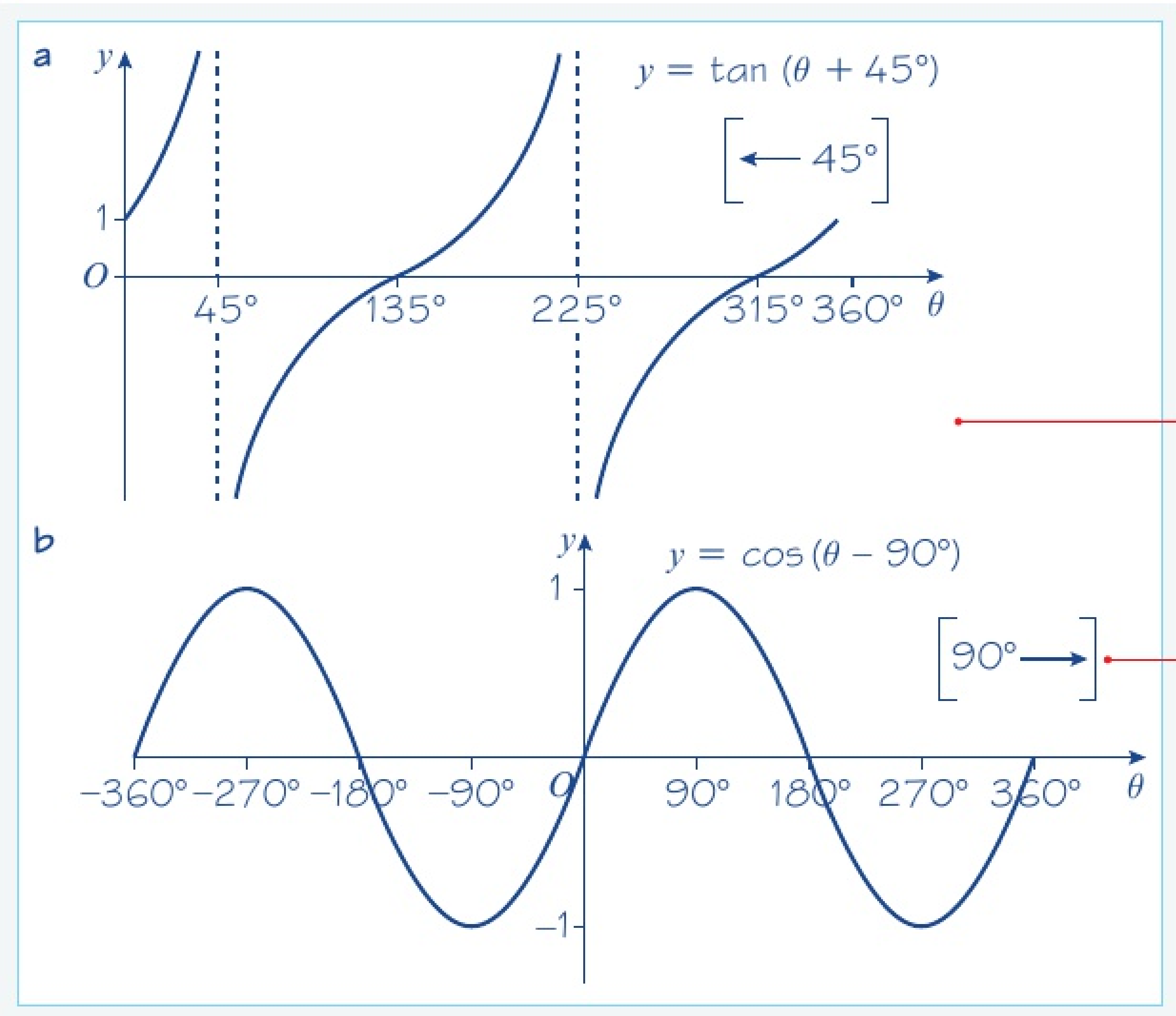
$y = f(x) + \frac{1}{2}$ is a translation of the graph $y = f(x)$ by vector $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$.

The graph of $y = \cos x$ is translated by $\frac{1}{2}$ unit in the positive y-direction.

Example 14 SKILLS INTERPRETATION

Sketch on separate sets of axes the graphs of:

a $y = \tan(\theta + 45°)$, $0 \leqslant \theta \leqslant 360°$ **b** $y = \cos(\theta - 90°)$, $-360° \leqslant \theta \leqslant 360°$



$y = f(\theta + 45°)$ is a translation of the graph $y = f(\theta)$ by vector $\begin{pmatrix} -45° \\ 0 \end{pmatrix}$. Remember to translate any asymptotes as well.

The graph of $y = \tan\theta$ is translated by 45° to the left. The asymptotes are now at $\theta = 45°$ and $\theta = 225°$. The curve meets the y-axis where $\theta = 0$, so $y = 1$.

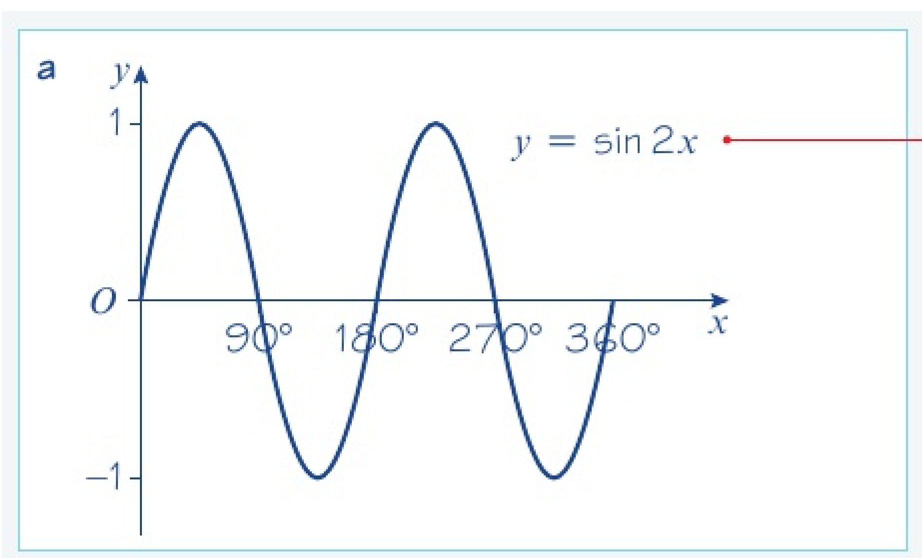
$y = f(\theta - 90°)$ is a translation of the graph $y = f(\theta)$ by vector $\begin{pmatrix} 90° \\ 0 \end{pmatrix}$.

The graph of $y = \cos\theta$ is translated by 90° to the right. Note that this is exactly the same curve as $y = \sin\theta$, so another property is that $\cos(\theta - 90°) = \sin\theta$.

Example 15

Sketch on separate sets of axes the graphs of:

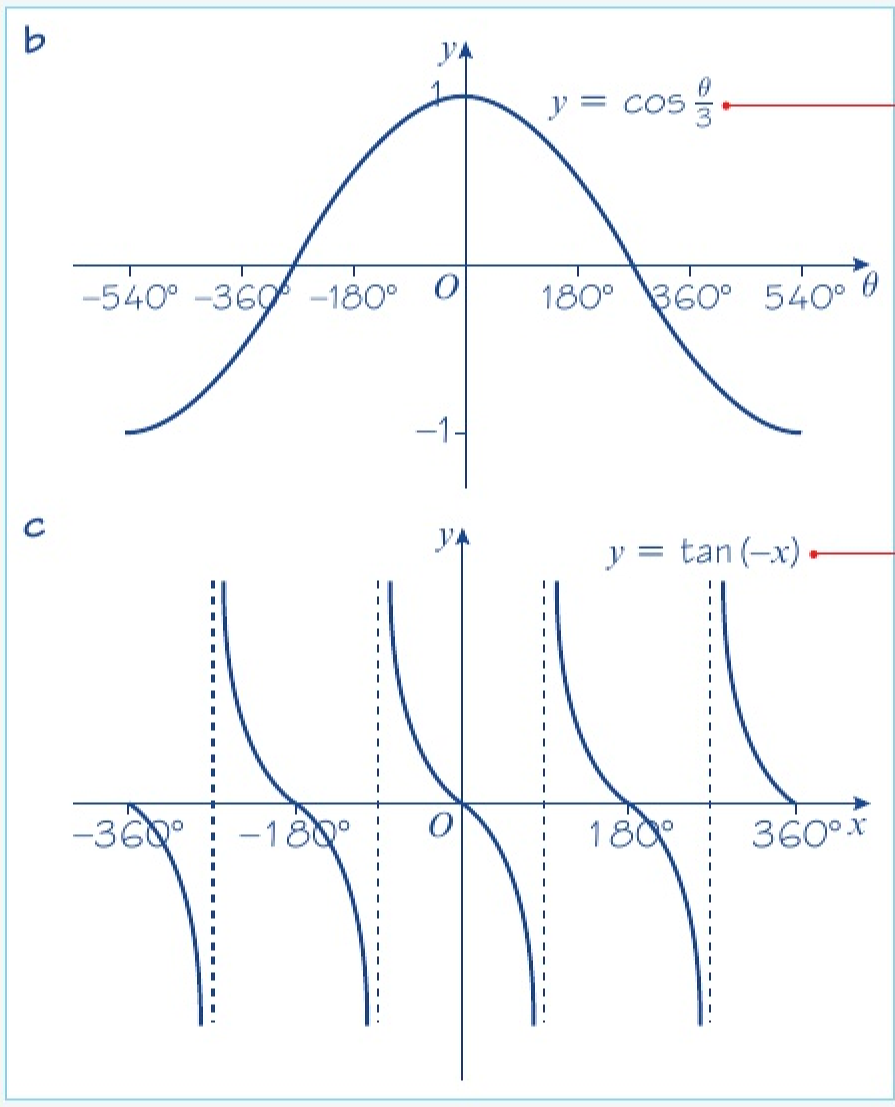
a $y = \sin 2x$, $0 \leqslant x \leqslant 360°$ **b** $y = \cos\frac{\theta}{3}$, $-540° \leqslant \theta \leqslant 540°$ **c** $y = \tan(-x)$, $-360° \leqslant x \leqslant 360°$



$y = f(2x)$ is a horizontal stretch of the graph $y = f(x)$ with scale factor $\frac{1}{2}$.

The graph of $y = \sin x$ is stretched horizontally with scale factor $\frac{1}{2}$.

The period is now 180° and two complete 'waves' are seen in the interval $0 \leqslant x \leqslant 360°$.



$y = f(\frac{1}{3}\theta)$ is a horizontal stretch of the graph $y = f(\theta)$ with scale factor 3.

The graph of $y = \cos\theta$ is stretched horizontally with scale factor 3. The period of $\cos\frac{\theta}{3}$ is 1080° and only one complete wave is seen while $-540 \leqslant \theta \leqslant 540°$. The curve crosses the θ-axis at $\theta = \pm 270°$.

$y = f(-x)$ is a reflection of the graph $y = f(x)$ in the y-axis.

The graph of $y = \tan(-x)$ is reflected in the y-axis. In this case the asymptotes are all vertical so they remain unchanged.

Online Plot transformations of trigonometric graphs using technology.

Exercise 6G SKILLS INTERPRETATION

1 Write down **i** the maximum value, and **ii** the minimum value, of the following expressions, and in each case give the smallest positive (or zero) value of x for which it occurs.

a $\cos x$ **b** $4\sin x$ **c** $\cos(-x)$

d $3 + \sin x$ **e** $-\sin x$ **f** $\sin 3x$

2 Sketch, on the same set of axes, in the interval $0 \leqslant \theta \leqslant 360°$, the graphs of $\cos\theta$ and $\cos 3\theta$.

3 Sketch, on separate sets of axes, the graphs of the following, in the interval $0 \leqslant \theta \leqslant 360°$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -\cos\theta$ **b** $y = \frac{1}{3}\sin\theta$ **c** $y = \sin\frac{1}{3}\theta$ **d** $y = \tan(\theta - 45°)$

4 Sketch, on separate sets of axes, the graphs of the following, in the interval $-180° \leqslant \theta \leqslant 180°$. Give the coordinates of points of intersection with the axes, and of maximum and minimum points where appropriate.

a $y = -2\sin\theta$ **b** $y = \tan(\theta + 180°)$ **c** $y = \cos 4\theta$ **d** $y = \sin(-\theta)$

5 Sketch, on separate sets of axes, the graphs of the following in the interval $-360° \leqslant \theta \leqslant 360°$. In each case give the periodicity of the function.

a $y = \sin\frac{1}{2}\theta$ **b** $y = -\frac{1}{2}\cos\theta$ **c** $y = \tan(\theta - 90°)$ **d** $y = \tan 2\theta$

(P) **6** **a** By considering the graphs of the functions, or otherwise, verify that:

i $\cos\theta = \cos(-\theta)$

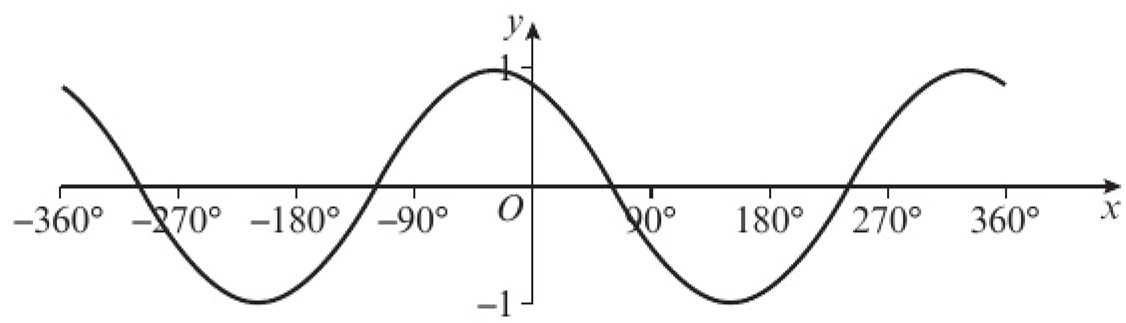
ii $\sin\theta = -\sin(-\theta)$

iii $\sin(\theta - 90°) = -\cos\theta$.

b Use the results in **a ii** and **iii** to show that $\sin(90° - \theta) = \cos\theta$.

c In Example 14 you saw that $\cos(\theta - 90°) = \sin\theta$.
Use this result with part **a i** to show that $\cos(90° - \theta) = \sin\theta$.

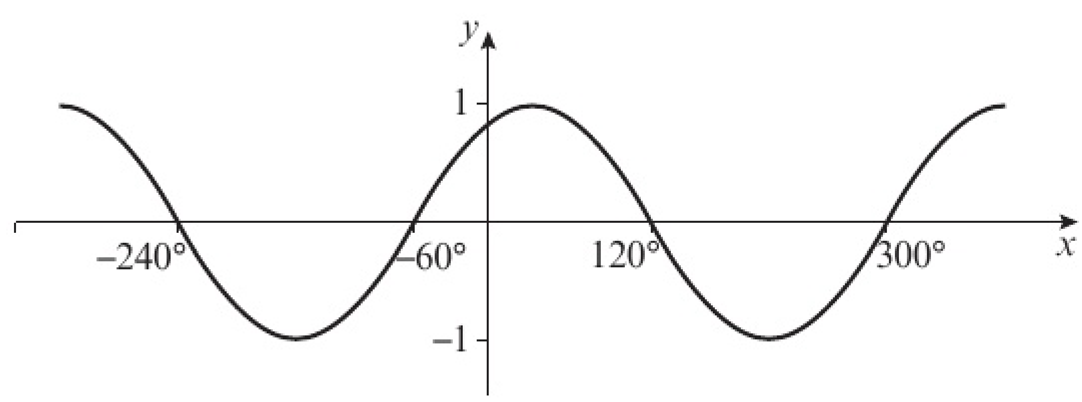
(E) **7** The graph shows the curve $y = \cos(x + 30°)$, $-360° \leqslant x \leqslant 360°$.



a Write down the coordinates of the points where the curve crosses the x-axis. **(2 marks)**

b Find the coordinates of the point where the curve crosses the y-axis. **(1 mark)**

(E/P) **8** The graph shows the curve with equation $y = \sin(x + k)$, $-360° \leqslant x \leqslant 360°$, where k is a constant.



a Find one possible value for k. **(2 marks)**

b Is there more than one possible answer to part **a**? Give a reason for your answer. **(2 marks)**

(E/P) **9** The variation in the depth of water in a rock pool can be modelled using the function $y = \sin(30t)$, where t is the time in hours and $0 \leqslant t \leqslant 6$.

a Sketch the function for the given interval. **(2 marks)**

b If $t = 0$ represents midday, during what times will the rock pool be at least half full? **(3 marks)**

Chapter review 6

SKILLS EXECUTIVE FUNCTION

Give non-exact answers to 3 significant figures.

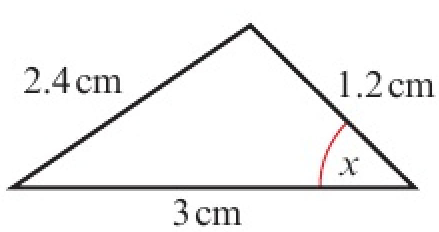
1 Triangle ABC has area 10 cm^2. $AB = 6$ cm, $BC = 8$ cm and $\angle ABC$ is obtuse. Find:

a the size of $\angle ABC$

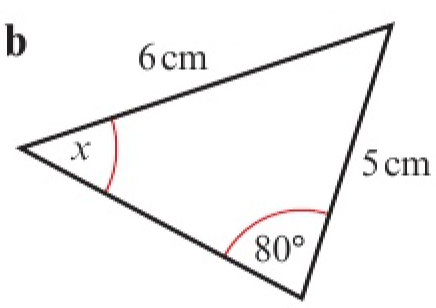
b the length of AC.

2 In each triangle below, find the size of x and the area of the triangle.

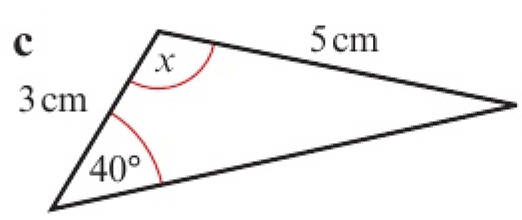
a



b



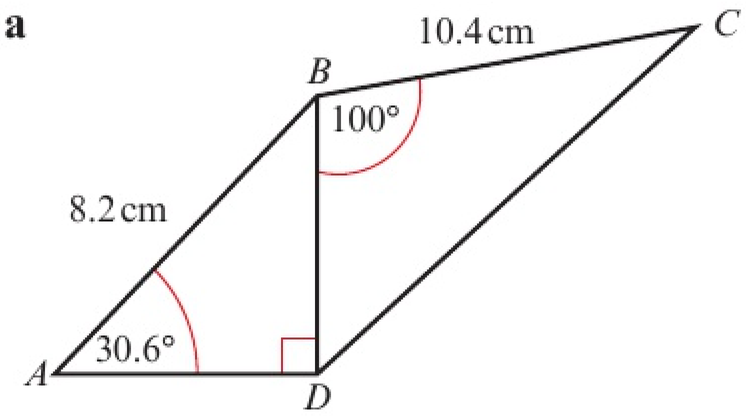
c



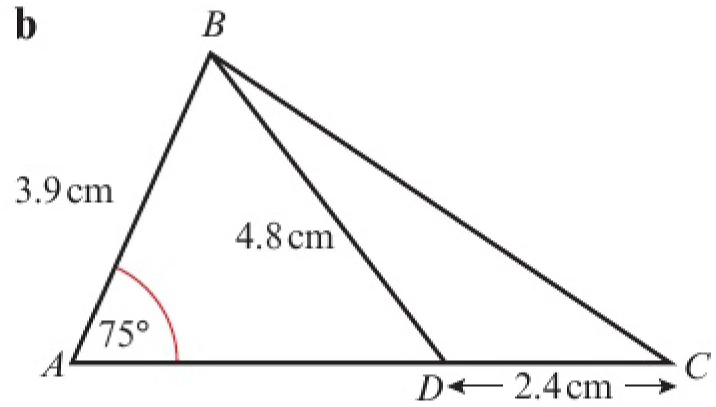
3 The sides of a triangle are 3 cm, 5 cm and 7 cm respectively.
Show that the largest angle is 120°, and find the area of the triangle.

(P) 4 Calculate the total area in each of the figures below.

a



b



5 In $\triangle ABC$, $AB = 10$ cm, $BC = a\sqrt{3}$ cm, $AC = 5\sqrt{13}$ cm and $\angle ABC = 150°$. Calculate:

a the value of a

b the exact area of $\triangle ABC$.

(P) 6 In a triangle, the largest side has length 2 cm and one of the other sides has length $\sqrt{2}$ cm. Given that the area of the triangle is 1 cm^2, show that the triangle is right-angled and isosceles.

(E/P) 7 The three points A, B and C, with coordinates $A(0, 1)$, $B(3, 4)$ and $C(1, 3)$ respectively, are joined to form a triangle.

a Show that $\cos \angle ACB = -\frac{4}{5}$. **(5 marks)**

b Calculate the area of $\triangle ABC$. **(2 marks)**

(E/P) 8 The longest side of a triangle has length $(2x - 1)$ cm. The other sides have lengths $(x - 1)$ cm and $(x + 1)$ cm. Given that the largest angle is 120°, work out:

a the value of x **(5 marks)**

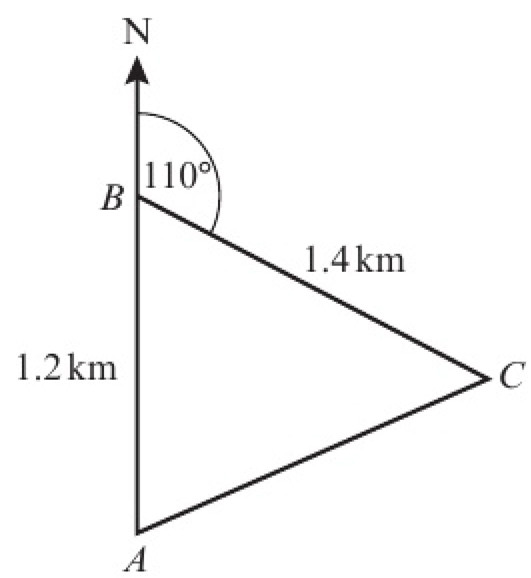
b the area of the triangle. **(3 marks)**

(E/P) 9 A park keeper walks 1.2 km due north from his hut at point A to point B. He then walks 1.4 km on a bearing of 110° from point B to point C.

a Find how far he is from his hut when at point C. Give your answer in km to 3 s.f. **(3 marks)**

b Work out the bearing of the hut from point C. Give your answer to the nearest degree. **(3 marks)**

c Work out the area enclosed by his walk. **(3 marks)**



(E/P) 10 A windmill has four identical triangular sails made from wood. If each triangle has sides of length 12 m, 15 m and 20 m, work out the total area of wood needed. **(5 marks)**

(E/P) 11 Two points, A and B are on level ground. A tower at point C has an angle of elevation from A of 15° and an angle of elevation from B of 32°. A and B are both on the same side of C, and A, B and C lie on the same straight line. The distance $AB = 75$ m.
Find the height of the tower. **(4 marks)**

12 Describe geometrically the transformations which map:

a the graph of $y = \tan x$ onto the graph of $\tan \frac{1}{2}x$

b the graph of $y = \tan \frac{1}{2}x$ onto the graph of $3 + \tan \frac{1}{2}x$

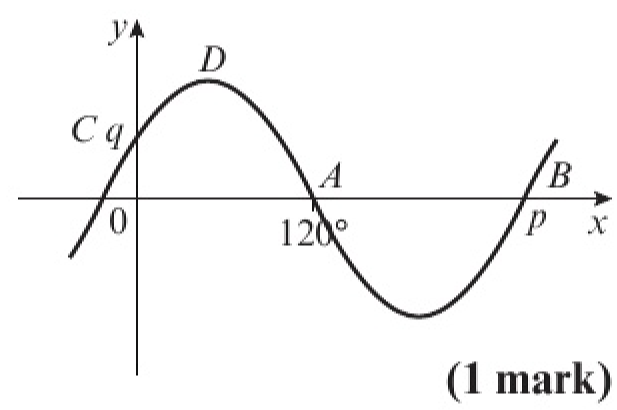
c the graph of $y = \cos x$ onto the graph of $-\cos x$

d the graph of $y = \sin(x - 10)$ onto the graph of $\sin(x + 10)$.

E/P **13** **a** Sketch on the same set of axes, in the interval $0 \leqslant x \leqslant 180°$, the graphs of $y = \tan(x - 45°)$ and $y = -2\cos x$, showing the coordinates of points of intersection with the axes. **(6 marks)**

b **Deduce** the number of solutions of the equation $\tan(x - 45°) + 2\cos x = 0$, in the interval $0 \leqslant x \leqslant 180°$. **(2 marks)**

E **14** The diagram shows part of the graph of $y = f(x)$. It crosses the x-axis at $A(120°, 0)$ and $B(p, 0)$. It crosses the y-axis at $C(0, q)$ and has a maximum value at D, as shown.



Given that $f(x) = \sin(x + k)$, where $k > 0$, write down

a the value of p **(1 mark)**

b the coordinates of D **(1 mark)**

c the smallest value of k **(1 mark)**

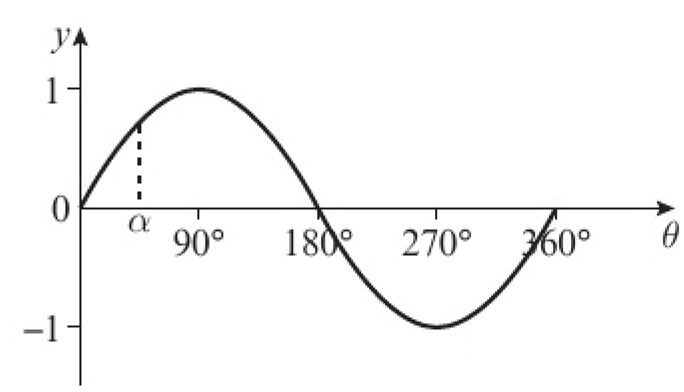
d the value of q. **(1 mark)**

E/P **15** Consider the function $f(x) = \sin px$, $p \in \mathbb{R}$, $0 \leqslant x \leqslant 360°$.
The closest point to the origin that the graph of $f(x)$ crosses the x-axis has x-coordinate 36°.

a Determine the value of p and sketch the graph of $y = f(x)$. **(5 marks)**

b Write down the period of $f(x)$. **(1 mark)**

16 The graph shows $y = \sin\theta$, $0 \leqslant \theta \leqslant 360°$, with one value of θ ($\theta = \alpha$) marked on the axis.



a Copy the graph and mark on the θ-axis the positions of $180° - \alpha$, $180° + \alpha$, and $360° - \alpha$.

b Verify that:
$\sin\alpha = \sin(180° - \alpha) = -\sin(180° + \alpha) = -\sin(360° - \alpha)$.

17 **a** Sketch on separate sets of axes the graphs of $y = \cos\theta$, $0 \leqslant \theta \leqslant 360°$, and $y = \tan\theta$, $0 \leqslant \theta \leqslant 360°$, and on each θ-axis mark the point $(\alpha, 0)$ as in question **16**.

b Verify that:

i $\cos\alpha = -\cos(180° - \alpha) = -\cos(180° + \alpha) = \cos(360° - \alpha)$

ii $\tan\alpha = -\tan(180° - \alpha) = \tan(180° + \alpha) = -\tan(360° - \alpha)$

E/P **18** A series of sand dunes has a cross-section which can be modelled using a sine curve of the form $y = \sin(60x°)$ where x is the length of the series of dunes in metres.

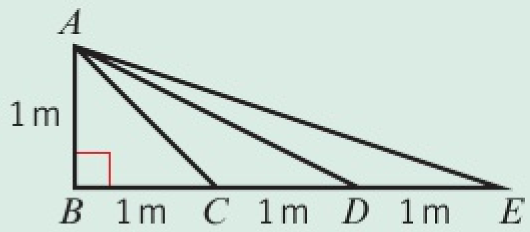
a Draw the graph of $y = \sin(60x°)$ for $0 \leqslant x \leqslant 24°$. **(3 marks)**

b Write down the number of sand dunes in this model. **(1 mark)**

c Give one reason why this may not be a realistic model. **(1 mark)**

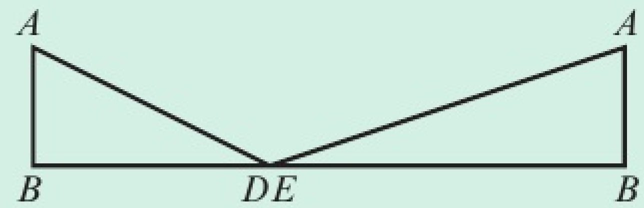
Challenge

In this diagram $AB = BC = CD = DE = 1$ m.



Prove that $\angle AEB + \angle ADB = \angle ACB$.

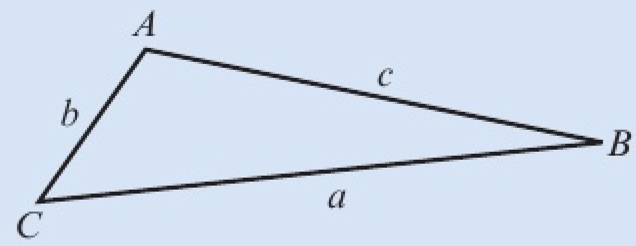
Hint Try drawing triangles ADB and AEB back to back.



Summary of key points

1 This version of the cosine rule is used to find a missing side if you know two sides and the angle between them:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



2 This version of the cosine rule is used to find an angle if you know all three sides:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

3 This version of the sine rule is used to find the length of a missing side:

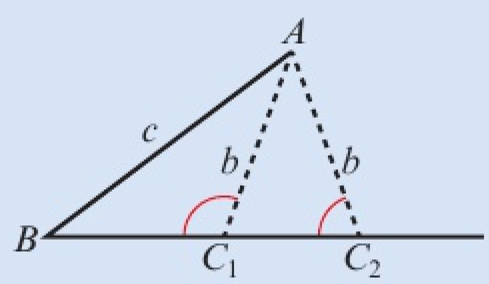
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

4 This version of the sine rule is used to find a missing angle:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

5 The sine rule sometimes produces two possible solutions for a missing angle:

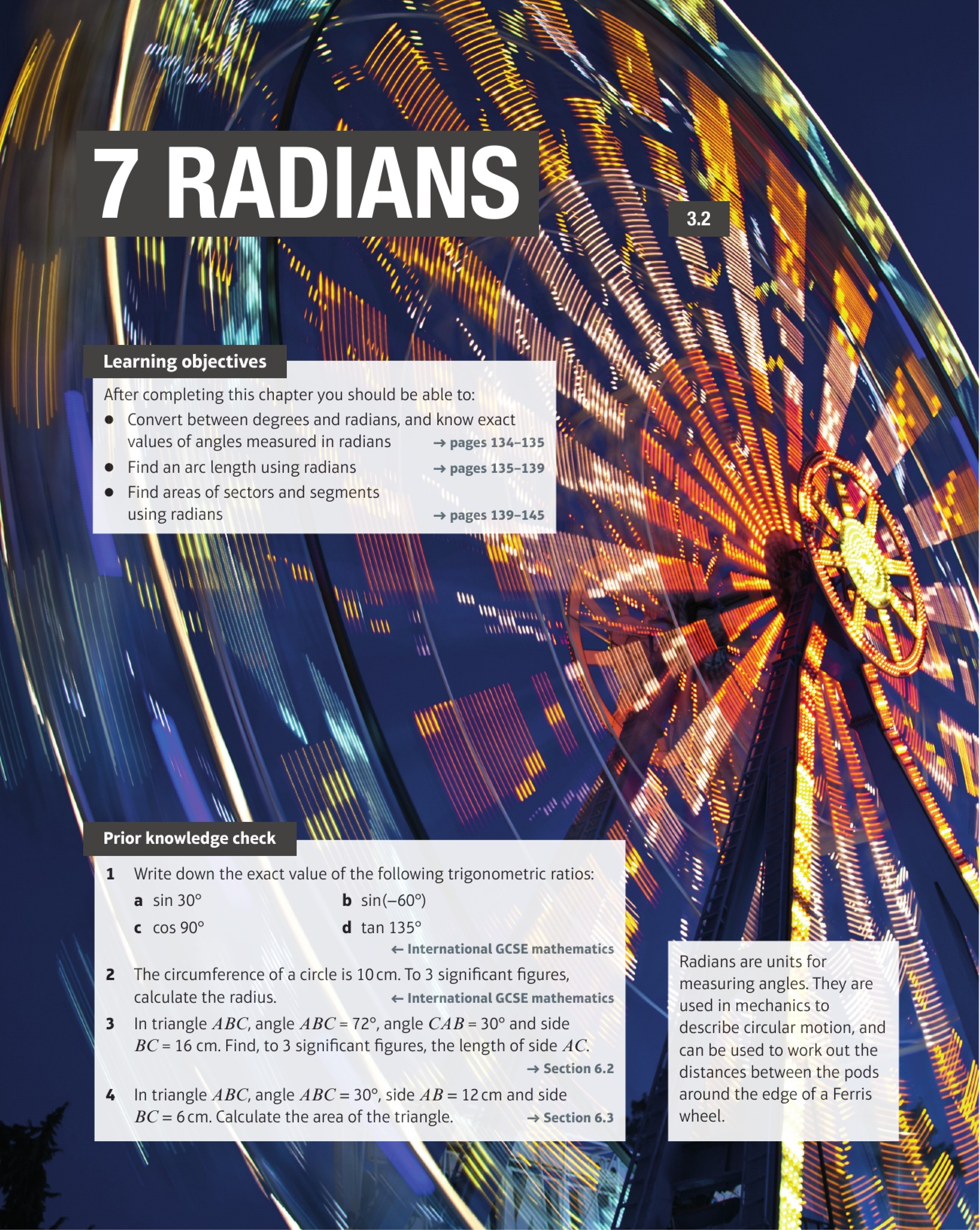
$$\sin \theta = \sin (180° - \theta)$$



6 Area of a triangle $= \frac{1}{2}ab \sin C$

7 The graphs of sine, cosine and tangent are periodic. They repeat themselves after a certain interval.

- The graph of $y = \sin \theta$ repeats itself every 360° period .
  It crosses the x-axis at …, −180°, 0°, 180°, 360°, …
  It has a maximum value of 1 and a minimum value of −1.
- The graph of $y = \cos \theta$ repeats itself every 360° period.
  It crosses the x-axis at …, −90°, 90°, 270°, 450°, …
  It has a maximum value of 1 and a minimum value of −1.
- The graph of $y = \tan \theta$ repeats itself every 180° period.
  It crosses the x-axis at … −180°, 0°, 180°, 360°, …
  It has no maximum or minimum value.
  It has vertical asymptotes at $x = -90°$, $x = 90°$, $x = 270°$, …



7 RADIANS

3.2

Learning objectives

After completing this chapter you should be able to:

- Convert between degrees and radians, and know exact values of angles measured in radians → pages 134–135
- Find an arc length using radians → pages 135–139
- Find areas of sectors and segments using radians → pages 139–145

Prior knowledge check

1 Write down the exact value of the following trigonometric ratios:

a sin 30°   **b** sin(–60°)

c cos 90°   **d** tan 135°

← International GCSE mathematics

2 The circumference of a circle is 10 cm. To 3 significant figures, calculate the radius. ← International GCSE mathematics

3 In triangle ABC, angle $ABC = 72°$, angle $CAB = 30°$ and side $BC = 16$ cm. Find, to 3 significant figures, the length of side AC. → Section 6.2

4 In triangle ABC, angle $ABC = 30°$, side $AB = 12$ cm and side $BC = 6$ cm. Calculate the area of the triangle. → Section 6.3

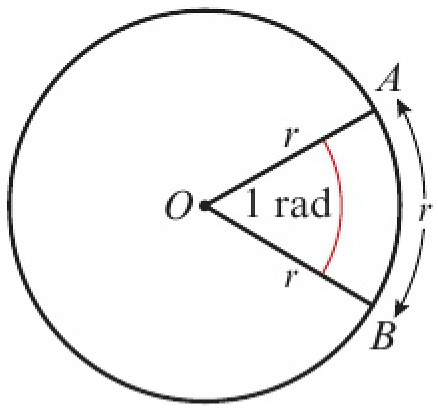
Radians are units for measuring angles. They are used in mechanics to describe circular motion, and can be used to work out the distances between the pods around the edge of a Ferris wheel.

7.1 Radian measure

So far you have probably only measured angles in degrees, with one degree representing $\frac{1}{360}$ of a complete revolution around a circle.

You can also measure angles in units called **radians**. 1 radian is the angle **subtended** at the centre of a circle by an **arc** whose length is equal to the radius of the circle.

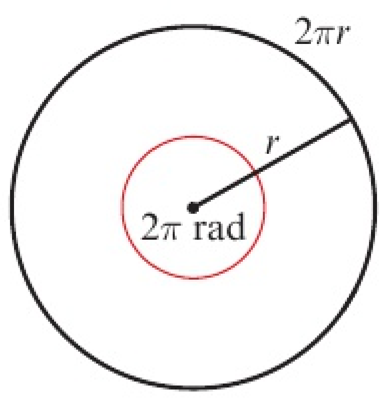
If the arc AB has length r, then $\angle AOB$ is 1 radian.



Notation You can write 1 radian as 1 rad or as 1^c.

The **circumference** of a circle of radius r is an arc of length $2\pi r$, so it subtends an angle of 2π radians at the centre of the circle.

- 2π radians = 360°
- π radians = 180°
- 1 radian = $\frac{180°}{\pi}$



Hint This means that 1 radian = 57.295...°

Example 1

Convert the following angles into degrees.

a $\frac{7\pi}{8}$ rad **b** $\frac{4\pi}{15}$ rad

a $\frac{7\pi}{8}$ rad

$= \frac{7}{8} \times 180°$

$= 157.5°$

b $\frac{4\pi}{15}$ rad

$= 4 \times \frac{180°}{15}$

$= 48°$

1 radian = $\frac{180°}{\pi}$, so multiply by $\frac{180°}{\pi}$:
$\frac{7\pi}{8} \times \frac{180°}{\pi} = \frac{7}{8} \times 180°$

Example 2

Convert the following angles into radians. Leave your answers in terms of π.

a 150° **b** 110°

a $150° = 150° \times \frac{\pi}{180°}$ rad

$= \frac{5\pi}{6}$ rad

b $110° = 110° \times \frac{\pi}{180°}$ rad

$= \frac{11\pi}{18}$ rad

$1° = \frac{\pi}{180°}$ radians, so multiply by $\frac{\pi}{180°}$

Your calculator will often give you exact answers in terms of π.

You should learn these important angles in radians:

- $30° = \frac{\pi}{6}$ radians
- $60° = \frac{\pi}{3}$ radians
- $180° = \pi$ radians
- $45° = \frac{\pi}{4}$ radians
- $90° = \frac{\pi}{2}$ radians
- $360° = 2\pi$ radians

Exercise 7A SKILLS INTERPRETATION

1 Without using a calculator, convert the following angles in radians to degrees:

a $\frac{\pi}{20}$ **b** $\frac{\pi}{15}$ **c** $\frac{5\pi}{12}$

d $\frac{\pi}{2}$ **e** $\frac{7\pi}{9}$ **f** $\frac{7\pi}{6}$

g $\frac{5\pi}{4}$ **h** $\frac{3\pi}{2}$ **i** 3π

2 Use your calculator to convert the following angles to degrees, giving your answer to the nearest 0.1°:

a 0.46^c **b** 1^c **c** 1.135^c **d** $\sqrt{3}^{\,c}$

e 2.5^c **f** 3.14^c **g** 3.49^c

3 Use your calculator to write down the value, to 3 significant figures, of the following trigonometric functions.

a $\sin 0.5^c$ **b** $\cos \sqrt{2}^{\,c}$ **c** $\tan 1.05^c$ **d** $\sin 2^c$ **e** $\cos 3.6^c$

4 Convert the following angles to radians, giving your answers as multiples of π in the form shown in Example 2:

a 8° **b** 10° **c** 22.5° **d** 30°

e 45° **f** 60° **g** 75° **h** 80°

i 112.5° **j** 120° **k** 135° **l** 200°

m 240° **n** 270° **o** 315° **p** 330°

5 Use your calculator to convert the following angles to radians, giving your answers to 3 significant figures:

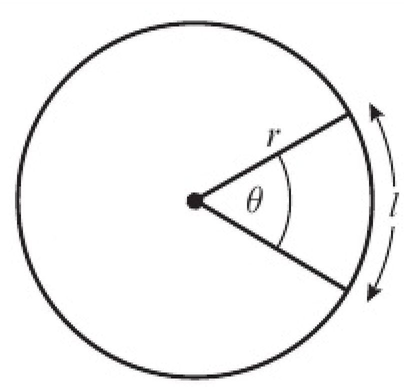
a 50° **b** 75° **c** 100°

d 160° **e** 230° **f** 320°

7.2 Arc length

Using radians greatly simplifies the formula for arc length.

- To find the arc length l of a **sector** of a circle, use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



Example 3

Find the length of the arc of a circle of radius 5.2 cm, given that the arc subtends an angle of 0.8 rad at the centre of the circle.

Online Explore the arc length of a sector using GeoGebra.

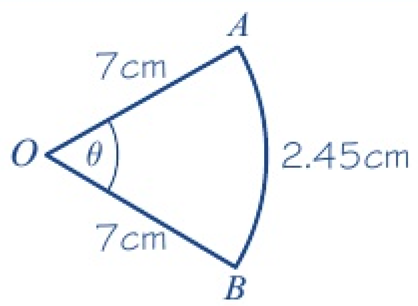
Arc length = 5.2 × 0.8 = 4.16 cm

Use $l = r\theta$, with $r = 5.2$ and $\theta = 0.8$.

Example 4

SKILLS ANALYSIS

An arc AB of a circle with radius 7 cm and centre O has a length of 2.45 cm. Find the angle $\angle AOB$ subtended by the arc at the centre of the circle.



$l = r\theta$

$2.45 = 7\theta$

$\frac{2.45}{7} = \theta$

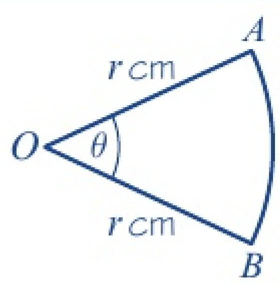
$\theta = 0.35$ rad

Use $l = r\theta$, with $l = 2.45$ and $r = 7$.

Using this formula gives the angle in radians.

Example 5

An arc AB of a circle, with centre O and radius r cm, subtends an angle of θ radians at O. The perimeter of the sector AOB is P cm. Express r in terms of P and θ.



$P = r\theta + 2r$

$= r(2 + \theta)$

So $r = \frac{P}{(2 + \theta)}$

Problem-solving

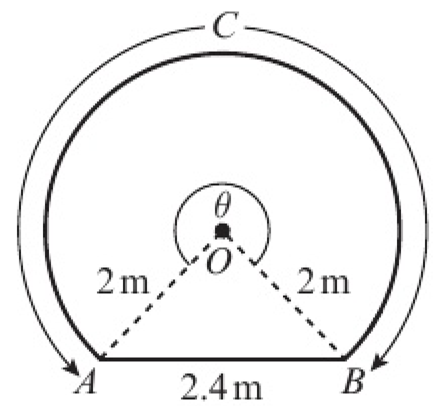
When given a problem in words, it is often a good idea to sketch and label a diagram to help you to visualise the information you have and what you need to find.

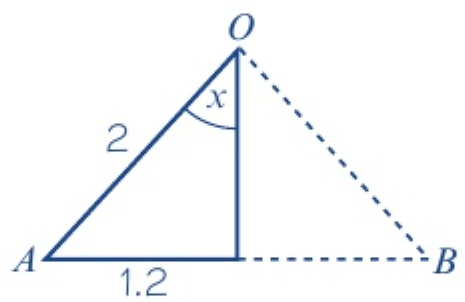
The perimeter = arc $AB + OA + OB$, where arc $AB = r\theta$.

Factorise.

Example 6 SKILLS ANALYSIS

The border of a garden pond consists of a straight edge AB of length 2.4 m, and a curved part C, as shown in the diagram. The curved part is an arc of a circle, centre O and radius 2 m. Find the length of C.





$\sin x = \frac{1.2}{2}$

$x = 0.6435\ldots$ rad

Acute $\angle AOB = 2x$ rad

$= 2 \times 0.6435\ldots$

$= 1.2870\ldots$ rad

So $\theta = (2\pi - 1.2870\ldots)$ rad

$= 4.9961\ldots$ rad

So length of $C = 9.99$ m (3 s.f.)

C subtends the reflex angle θ at O, so the length of $C = 2\theta$.

θ + acute $\angle AOB = 2\pi$ rad

$C = 2\theta$

Online Explore the area of a sector using GeoGebra.

Problem-solving Look for opportunities to use the basic trigonometric ratios rather than the more complicated cosine rule or sine rule. AOB is an isosceles triangle, so you can divide it into congruent right-angled triangles. Make sure your calculator is in radians mode.

Exercise 7B SKILLS ANALYSIS

1 An arc AB of a circle, centre O and radius r cm, subtends an angle θ radians at O. The length of AB is l cm.

a Find l when: **i** $r = 6, \theta = 0.45$ **ii** $r = 4.5, \theta = 0.45$ **iii** $r = 20, \theta = \frac{3}{8}\pi$

b Find r when: **i** $l = 10, \theta = 0.6$ **ii** $l = 1.26, \theta = 0.7$ **iii** $l = 1.5\pi, \theta = \frac{5}{12}\pi$

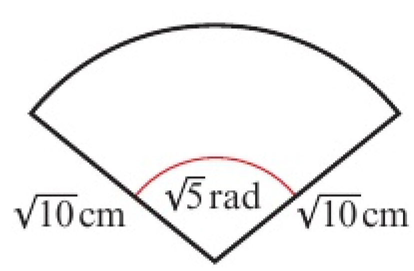
c Find θ when: **i** $l = 10, r = 7.5$ **ii** $l = 4.5, r = 5.625$ **iii** $l = \sqrt{12}, r = \sqrt{3}$

P 2 A **minor arc** AB of a circle, centre O and radius 10 cm, subtends an angle x at O. The **major arc** AB subtends an angle $5x$ at O. Find, in terms of π, the length of the minor arc AB.

Notation The **minor arc** AB is the shorter arc between points A and B on a circle.

3 An arc AB of a circle, centre O and radius 6 cm, has length l cm. Given that the **chord** AB has length 6 cm, find the value of l, giving your answer in terms of π.

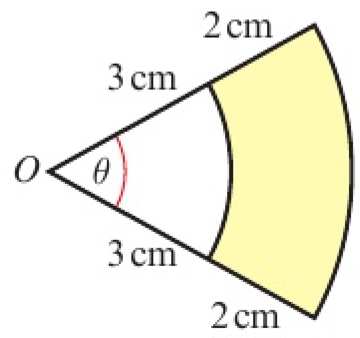
4 The sector of a circle of radius $\sqrt{10}$ cm contains an angle of $\sqrt{5}$ radians, as shown in the diagram. Find the length of the arc, giving your answer in the form $p\sqrt{q}$ cm, where p and q are integers.



(P) **5** Referring to the diagram, find:

a the perimeter of the shaded region when $\theta = 0.8$ radians

b the value of θ when the perimeter of the shaded region is 14 cm.



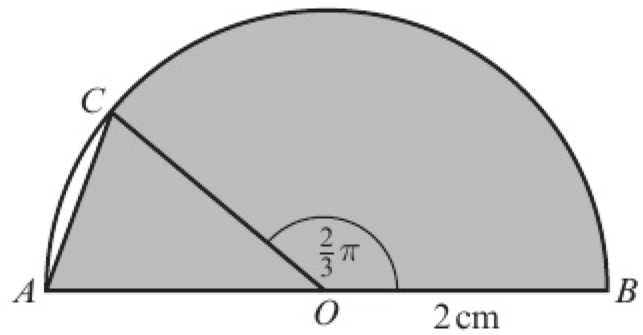
Problem-solving

The radius of the larger arc is 3 + 2 = 5 cm.

(P) **6** A sector of a circle of radius r cm contains an angle of 1.2 radians. Given that the sector has the same perimeter as a square of area 36 cm^2, find the value of r.

(P) **7** A sector of a circle of radius 15 cm contains an angle of θ radians. Given that the perimeter of the sector is 42 cm, find the value of θ.

(E/P) **8** In the diagram AB is the diameter of a circle, centre O and radius 2 cm.
The point C is on the circumference such that $\angle COB = \frac{2}{3}\pi$ radians.



a State the value, in radians, of $\angle COA$. **(1 mark)**

The shaded region enclosed by the chord AC, arc CB and AB is the template for a brooch.

b Find the exact value of the perimeter of the brooch. **(5 marks)**

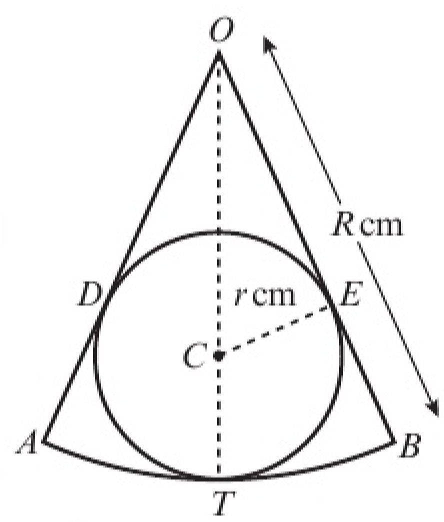
(P) **9** The points A and B lie on the circumference of a circle with centre O and radius 8.5 cm.
The point C lies on the major arc AB. Given that $\angle ACB = 0.4$ radians, calculate the length of the minor arc AB.

(E/P) **10** In the diagram OAB is a sector of a circle, centre O and radius R cm, and $\angle AOB = 2\theta$ radians. A circle, centre C and radius r cm, touches the arc AB at T, and touches OA and OB at D and E respectively, as shown.

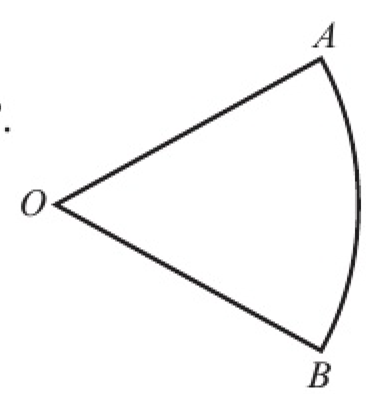
a Write down, in terms of R and r, the length of OC. **(1 mark)**

b Using $\triangle OCE$, show that $R\sin\theta = r(1 + \sin\theta)$. **(3 marks)**

c Given that $\sin\theta = \frac{3}{4}$ and that the perimeter of the sector OAB is 21 cm, find r, giving your answer to 3 significant figures. **(7 marks)**



(P) **11** The diagram shows a sector AOB.
The perimeter of the sector is twice the length of the arc AB.
Find the size of angle AOB.

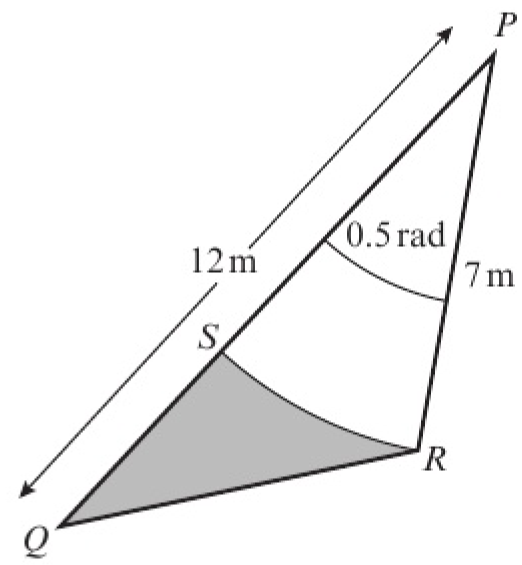


(P) **12** A circular Ferris wheel has 24 pods equally spaced on its circumference.

a Given the arc length between each pod is $\frac{3\pi}{2}$ m, and modelling each pod as a particle, calculate the diameter of the Ferris wheel.

b Given that it takes approximately 30 seconds for a pod to complete one revolution, estimate the speed of the pod in km/h.

E/P **13** The diagram shows a triangular garden, PQR, with $PQ = 12\,\text{m}$, $PR = 7\,\text{m}$ and $\angle QPR = 0.5$ radians. The curve SR is a small path separating the shaded patio area and the lawn, and is an arc of a circle with centre at P and radius 7 m.

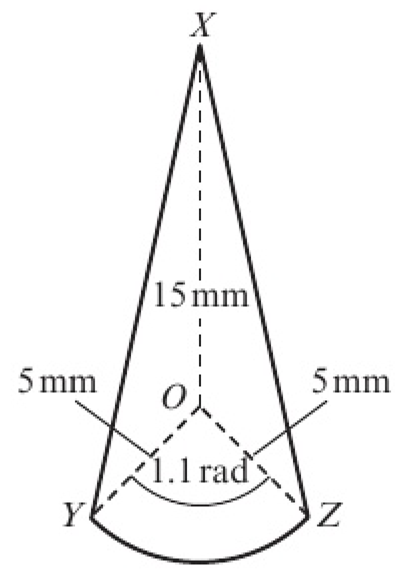


Find:

a the length of the path SR **(2 marks)**

b the perimeter of the shaded patio, giving your answer to 3 significant figures. **(4 marks)**

E/P **14** The shape XYZ shown is a design for an earring.



The straight lines XY and XZ are equal in length. The curve YZ is an arc of a circle with centre O and radius 5 mm. The size of $\angle YOZ$ is 1.1 radians and $XO = 15\,\text{mm}$.

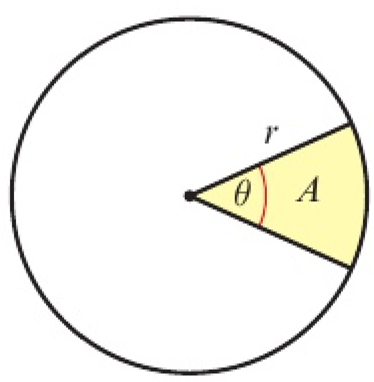
a Find the size of $\angle XOZ$, in radians, to 3 significant figures. **(2 marks)**

b Find the total perimeter of the earring, to the nearest mm. **(6 marks)**

7.3 Areas of sectors and segments

Using radians also greatly simplifies the formula for the area of a **sector**.

- To find the area A of a sector of a circle, use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



Notation A sector of a circle is the portion of a circle enclosed by two **radii** and an arc. The smaller area is known as the **minor** sector and the larger is known as the **major** sector.

Example 7

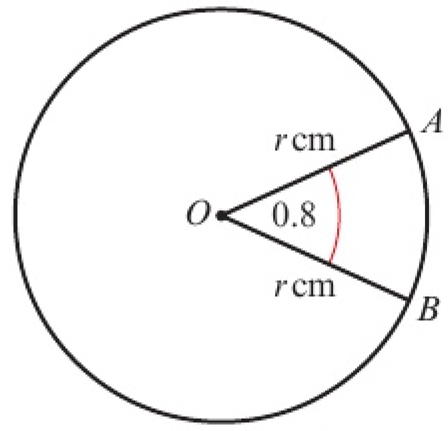
Find the area of the sector of a circle of radius 2.44 cm, given that the sector subtends an angle of 1.4 radians at the centre of the circle.

$$\text{Area of sector} = \frac{1}{2} \times 2.44^2 \times 1.4$$
$$= 4.17\,\text{cm}^2 \text{ (3 s.f.)}$$

Use $A = \frac{1}{2}r^2\theta$ with $r = 2.44$ and $\theta = 1.4$.

Example 8

In the diagram, the area of the minor sector AOB is $28.9\,\text{cm}^2$. Given that $\angle AOB = 0.8$ radians, calculate the value of r.



$28.9 = \frac{1}{2}r^2 \times 0.8 = 0.4r^2$

So $r^2 = \frac{28.9}{0.4} = 72.25$

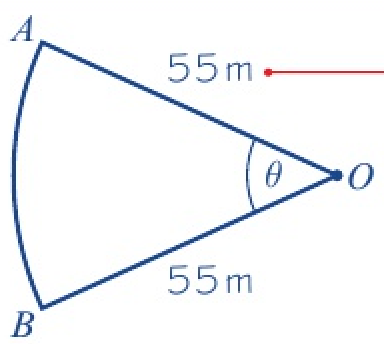
$r = \sqrt{72.25} = 8.5$

Let area of sector be $A\,\text{cm}^2$, and use $A = \frac{1}{2}r^2\theta$.

Use the positive square root in this case as a length cannot be negative.

Example 9

A plot of land is in the shape of a sector of a circle of radius 55 m. The length of fencing that is erected along the edge of the plot to enclose the land is 176 m. Calculate the area of the plot of land.



Arc $AB = 176 - (55 + 55)$

$= 66\,\text{m}$

$66 = 55\theta$

So $\theta = 1.2$ radians

Area of plot $= \frac{1}{2} \times 55^2 \times 1.2$

$= 1815\,\text{m}^2$

Online Explore the area of a segment using GeoGebra.

Draw a diagram including all the data and let the angle of the sector be θ.

Problem-solving

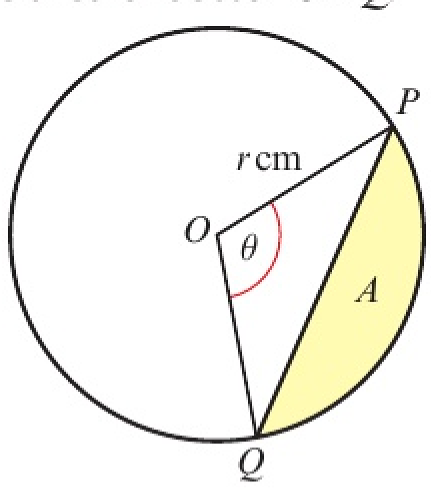
In order to find the area of the sector, you need to know θ. Use the information about the perimeter to find the arc length AB.

As the perimeter is given, first find length of arc AB.

Use the formula for arc length, $l = r\theta$.

Use the formula for area of a sector, $A = \frac{1}{2}r^2\theta$.

You can find the area of a **segment** by subtracting the area of triangle OPQ from the area of sector OPQ.



Using $\frac{1}{2}r^2\theta$ for the area of the sector and $\frac{1}{2}ab\sin\theta$ for the area of the triangle:

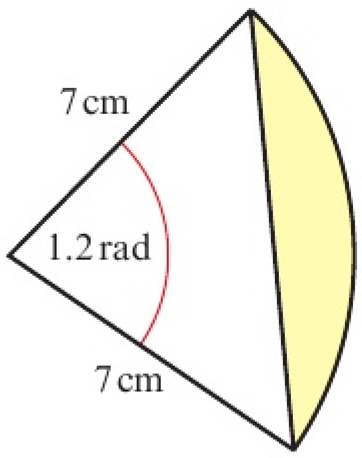
$A = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$= \frac{1}{2}r^2(\theta - \sin\theta)$

- The area of a segment in a circle of radius r is $A = \frac{1}{2}r^2(\theta - \sin\theta)$

Example 10 SKILLS ANALYSIS

The diagram shows a sector of a circle. Find the area of the shaded segment.



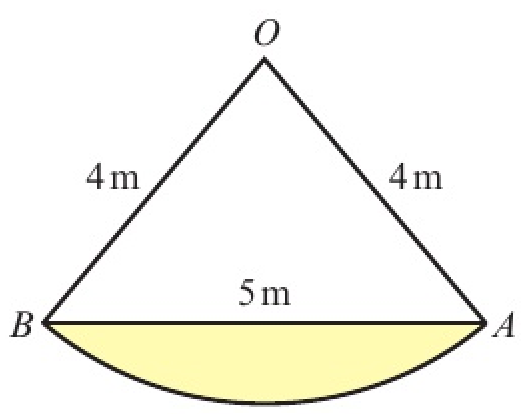
Area of segment $= \frac{1}{2} \times 7^2(1.2 - \sin 1.2)$

$= \frac{1}{2} \times 49 \times 0.26796...$

$= 6.57\text{ cm}^2$ (3 s.f.)

Use $A = \frac{1}{2}r^2(\theta - \sin\theta)$ with $r = 7$ and $\theta = 1.2$ radians. Make sure your calculator is in radians mode when calculating $\sin\theta$.

Example 11



In the diagram above, OAB is a sector of a circle, radius 4 m. The chord AB is 5 m long. Find the area of the shaded segment.

Calculate angle AOB first:

$\cos\angle AOB = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$

$= \frac{7}{32}$

So $\angle AOB = 1.3502...$

Area of shaded segment

$= \frac{1}{2} \times 4^2(1.3502... - \sin 1.3502...)$

$= \frac{1}{2} \times 16 \times 0.37448...$

$= 3.00\text{ m}^2$ (3 s.f.)

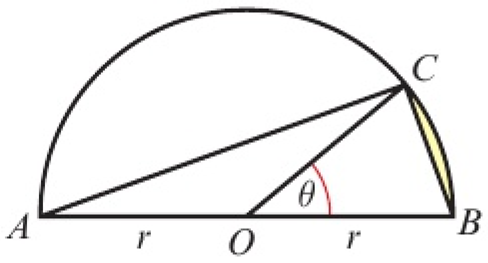
Problem-solving In order to find the area of the segment you need to know angle AOB. You can use the cosine rule in triangle AOB, or divide the triangle into two right-angled triangles and use the trigonometric ratios.

Use the cosine rule for a non-right-angled triangle.

Watch out Use unrounded values in your calculations wherever possible to avoid rounding errors. You can use the memory function or answer button on your calculator.

Example 12

In the diagram, AB is the diameter of a circle of radius r cm, and $\angle BOC = \theta$ radians.
Given that the area of $\triangle AOC$ is three times that of the shaded segment, show that $3\theta - 4\sin\theta = 0$.



Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

Area of $\triangle AOC = \frac{1}{2}r^2\sin(\pi - \theta)$

$= \frac{1}{2}r^2\sin\theta$

So $\quad \frac{1}{2}r^2\sin\theta = 3 \times \frac{1}{2}r^2(\theta - \sin\theta)$

$\sin\theta = 3(\theta - \sin\theta)$

So $\quad 3\theta - 4\sin\theta = 0$

Area of segment = area of sector – area of triangle

$\angle AOB = \pi$ radians

Area of $\Delta AOC = 3 \times$ area of shaded segment

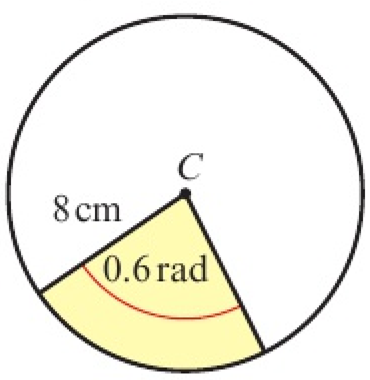
Problem-solving

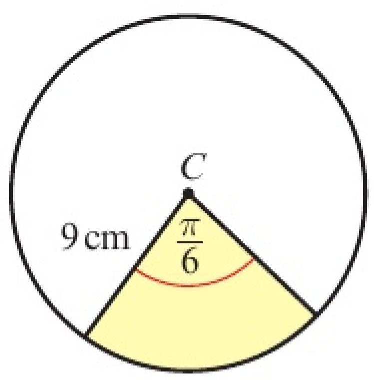
You might need to use circle theorems or properties when solving problems. The angle in a semicircle is a right angle so $\angle ACB = \frac{\pi}{2}$.

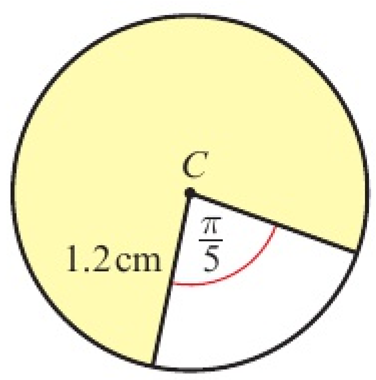
Exercise 7C

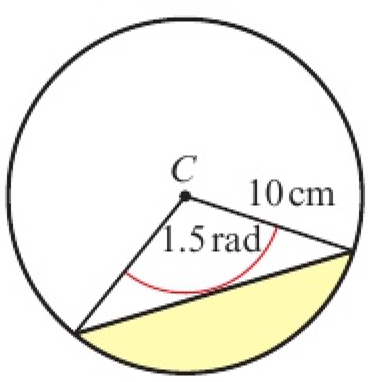
SKILLS ANALYSIS

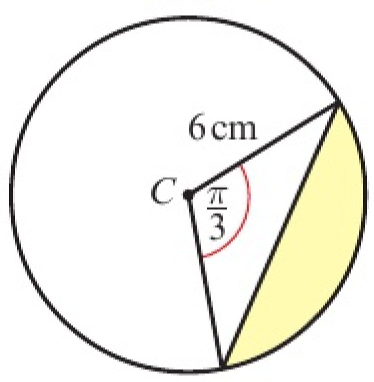
1 Find the shaded area in each of the following circles.
Leave your answers in terms of π where appropriate.

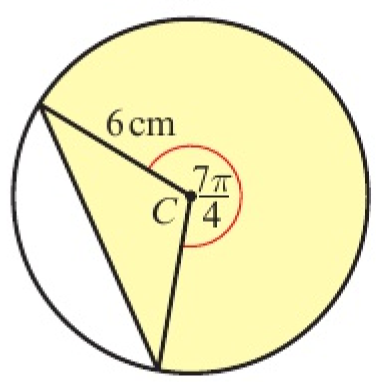
a


b


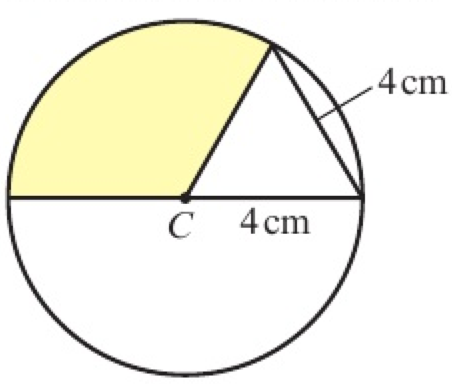
c


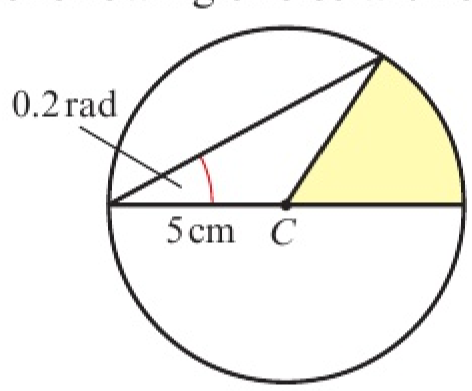
d


e


f


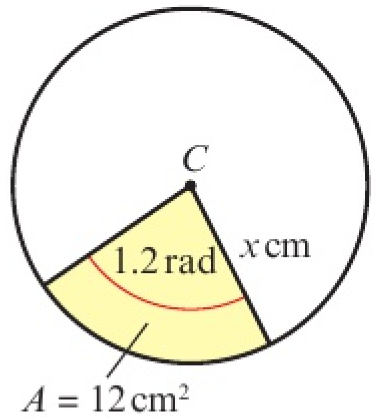
2 Find the shaded area in each of the following circles with centre C.

a


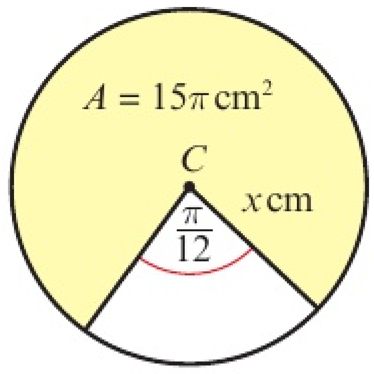
b


3 For the following circles with centre C, the area A of the shaded sector is given. Find the value of x in each case.

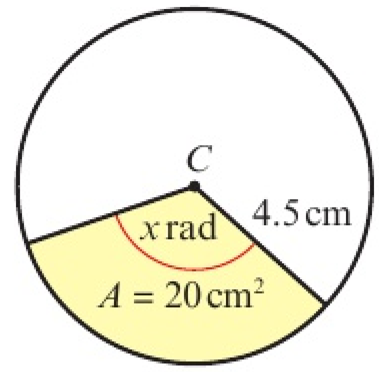
a



b



c



4 The arc AB of a circle, centre O and radius 6 cm, has length 4 cm. Find the area of the minor sector AOB.

5 The chord AB of a circle, centre O and radius 10 cm, has length 18.65 cm and subtends an angle of θ radians at O.

a Show that $\cos\theta = -0.739$ (to 3 significant figures).

b Find the area of the minor sector AOB.

(P) **6** The area of a sector of a circle of radius 12 cm is 100 cm^2. Find the perimeter of the sector.

7 The arc AB of a circle, centre O and radius r cm, is such that $\angle AOB = 0.5$ radians. Given that the perimeter of the minor sector AOB is 30 cm,

a calculate the value of r

b show that the area of the minor sector AOB is 36 cm^2

c calculate the area of the segment enclosed by the chord AB and the minor arc AB.

(P) **8** The arc AB of a circle, centre O and radius x cm, is such that angle $AOB = \frac{\pi}{12}$ radians. Given that the arc length AB is l cm,

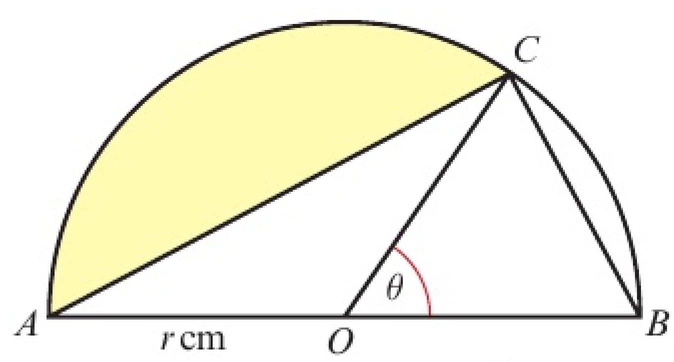
a show that the area of the sector can be written as $\frac{6l^2}{\pi}$.

The area of the full circle is 3600π cm^2.

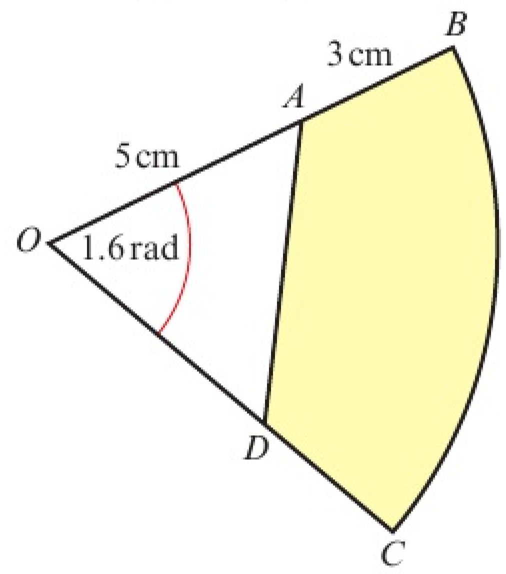
b Find the arc length of AB.

c Calculate the value of x.

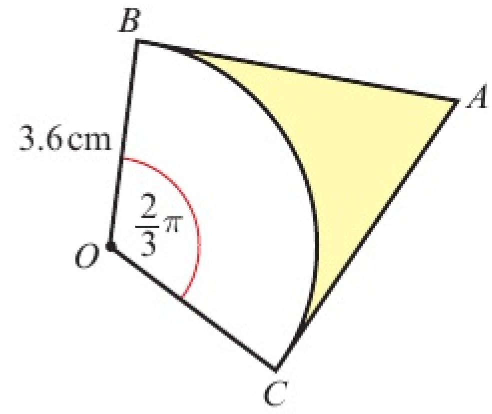
(P) **9** In the diagram, AB is the diameter of a circle of radius r cm and $\angle BOC = \theta$ radians. Given that the area of ΔCOB is equal to that of the shaded segment, show that $\theta + 2\sin\theta = \pi$.



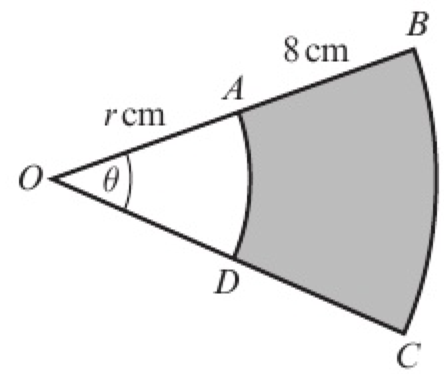
(P) **10** In the diagram, BC is the arc of a circle, centre O and radius 8 cm. The points A and D are such that $OA = OD = 5$ cm. Given that $\angle BOC = 1.6$ radians, calculate the area of the shaded region.



(P) **11** In the diagram, AB and AC are tangents to a circle, centre O and radius 3.6 cm. Calculate the area of the shaded region, given that $\angle BOC = \frac{2\pi}{3}$ radians.



(E/P) **12** In the diagram, AD and BC are arcs of circles with centre O, such that $OA = OD = r$ cm, $AB = DC = 8$ cm and $\angle BOC = \theta$ radians.



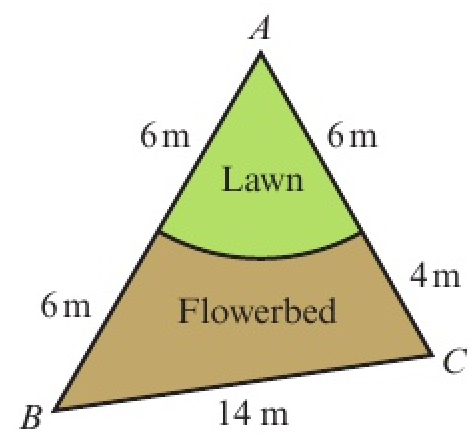
a Given that the area of the shaded region is 48 cm^2, show that

$$r = \frac{6}{\theta} - 4.$$ **(4 marks)**

b Given also that $r = 10\theta$, calculate the perimeter of the shaded region. **(6 marks)**

(P) **13** A sector of a circle of radius 28 cm has perimeter P cm and area A cm^2. Given that $A = 4P$, find the value of P.

(P) **14** The diagram shows a triangular plot of land. The sides AB, BC and CA have lengths 12 m, 14 m and 10 m respectively. The lawn is a sector of a circle, centre A and radius 6 m.

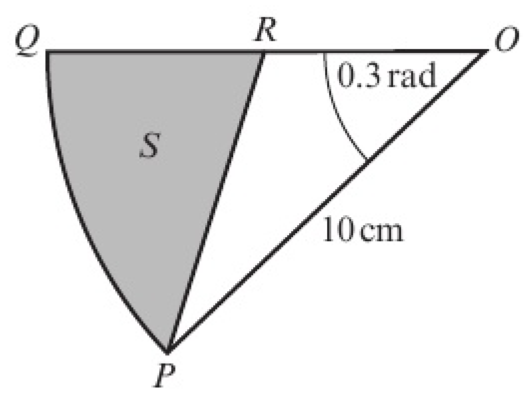


a Show that $\angle BAC = 1.37$ radians, correct to 3 significant figures.

b Calculate the area of the flowerbed.

(E/P) **15** The diagram shows OPQ, a sector of a circle with centre O, radius 10 cm, with $\angle POQ = 0.3$ radians.

The point R is on OQ such that the ratio $OR : RQ$ is $1:3$. The region S, shown shaded in the diagram, is **bounded** by QR, RP and the arc PQ.



Find:

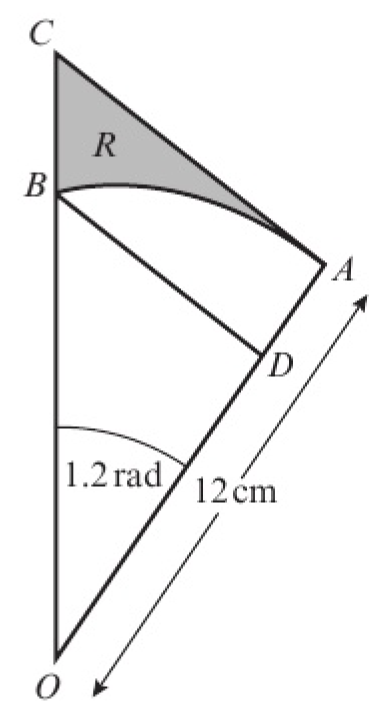
a the perimeter of S, giving your answer to 3 significant figures **(6 marks)**

b the area of S, giving your answer to 3 significant figures. **(6 marks)**

(E/P) **16** The diagram shows the sector OAB of a circle with centre O, radius 12 cm and angle 1.2 radians.

The line AC is a tangent to the circle with centre O, and OBC is a straight line.

The region R is bounded by the arc AB and the lines AC and CB.

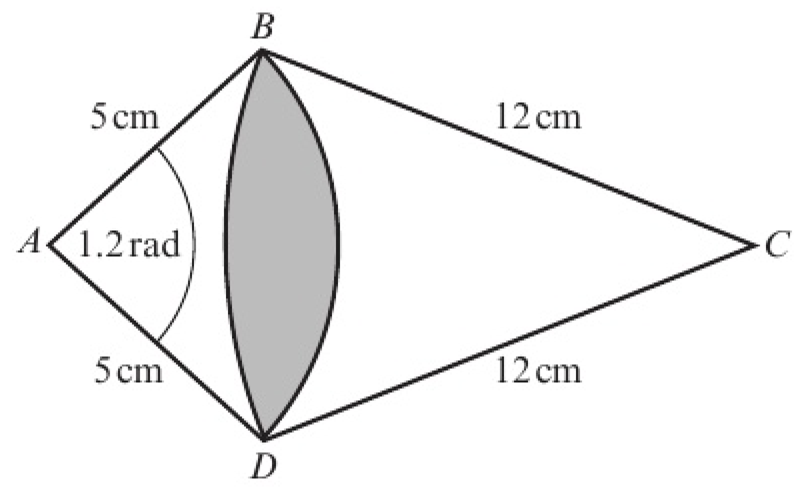


a Find the area of R, giving your answer to 2 decimal places. **(8 marks)**

The line BD is parallel to AC.

b Find the perimeter of DAB. **(5 marks)**

P **17** The diagram shows two intersecting sectors: ABD, with radius 5 cm and angle 1.2 radians, and CBD, with radius 12 cm.
Find the area of the overlapping section.



Challenge

Find an expression for the area of a sector of a circle with radius r and arc length l.

Chapter review 7

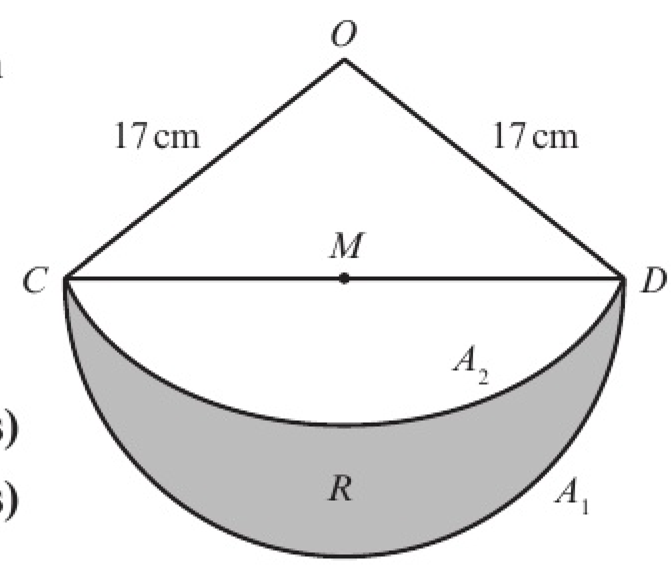
SKILLS EXECUTIVE FUNCTION

P **1** Triangle ABC is such that $AB = 5$ cm, $AC = 10$ cm and $\angle ABC = 90°$.
An arc of a circle, centre A and radius 5 cm, cuts AC at D.

a State, in radians, the value of $\angle BAC$.

b Calculate the area of the region enclosed by BC, DC and the arc BD.

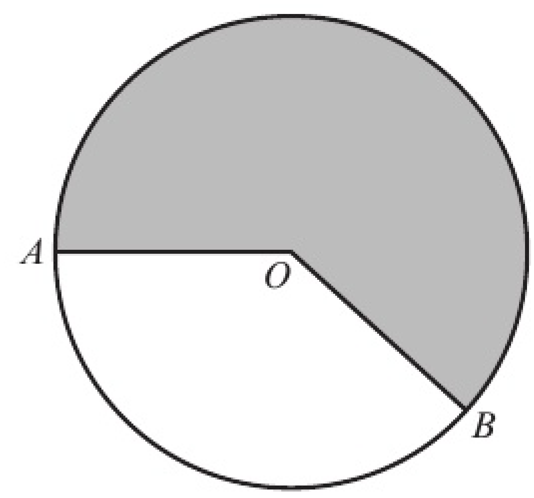
E/P **2** The diagram shows the triangle OCD with $OC = OD = 17$ cm and $CD = 30$ cm. The midpoint of CD is M. A semicircular arc A_1, with centre M is drawn, with CD as diameter.
A circular arc A_2 with centre O and radius 17 cm, is drawn from C to D. The shaded region R is bounded by the arcs A_1 and A_2. Calculate, giving answers to 2 decimal places:



a the area of the triangle OCD **(4 marks)**

b the area of the shaded region R. **(5 marks)**

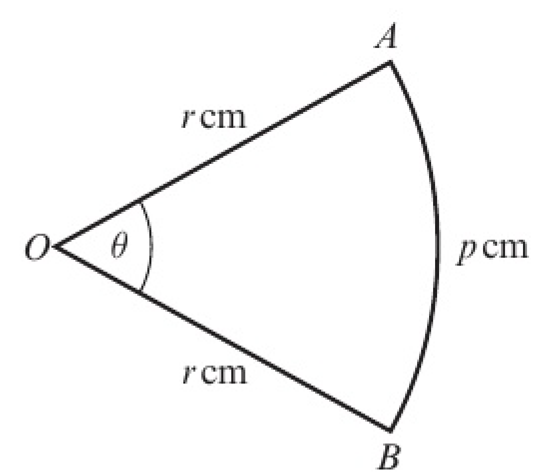
E/P **3** The diagram shows a circle, centre O, of radius 6 cm.
The points A and B are on the circumference of the circle.
The area of the shaded major sector is 80 cm^2.
Given that $\angle AOB = \theta$ radians, where $0 < \theta < \pi$, calculate:



a the value, to 3 decimal places, of θ **(3 marks)**

b the length in cm, to 2 decimal places, of the minor arc AB. **(2 marks)**

E/P **4** The diagram shows a sector OAB of a circle, centre O and radius r cm. The length of the arc AB is p cm and $\angle AOB$ is θ radians.



a Find θ in terms of p and r. **(2 marks)**

b Deduce that the area of the sector is $\frac{1}{2}pr$ cm^2. **(2 marks)**

Given that $r = 4.7$ and $p = 5.3$, where each has been measured to 1 decimal place, find, giving your answer to 3 decimal places:

c the least possible value of the area of the sector **(2 marks)**

d the range of possible values of θ. **(3 marks)**

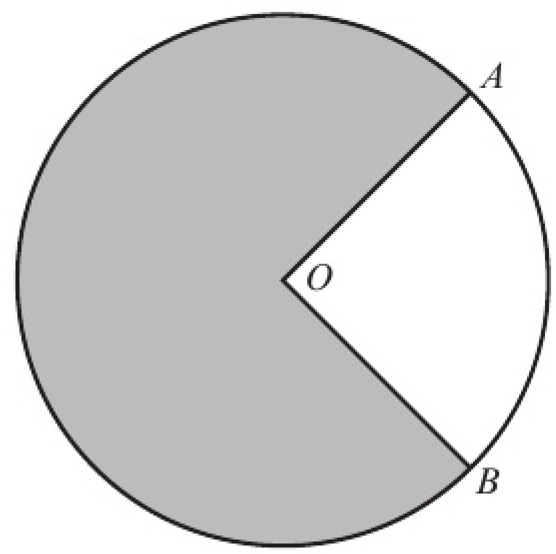
(E) **5** The diagram shows a circle centre O and radius 5 cm.
The length of the minor arc AB is 6.4 cm.

a Calculate, in radians, the size of the acute angle AOB. **(2 marks)**

The area of the minor sector AOB is R_1 cm^2 and the area of the shaded major sector is R_2 cm^2.

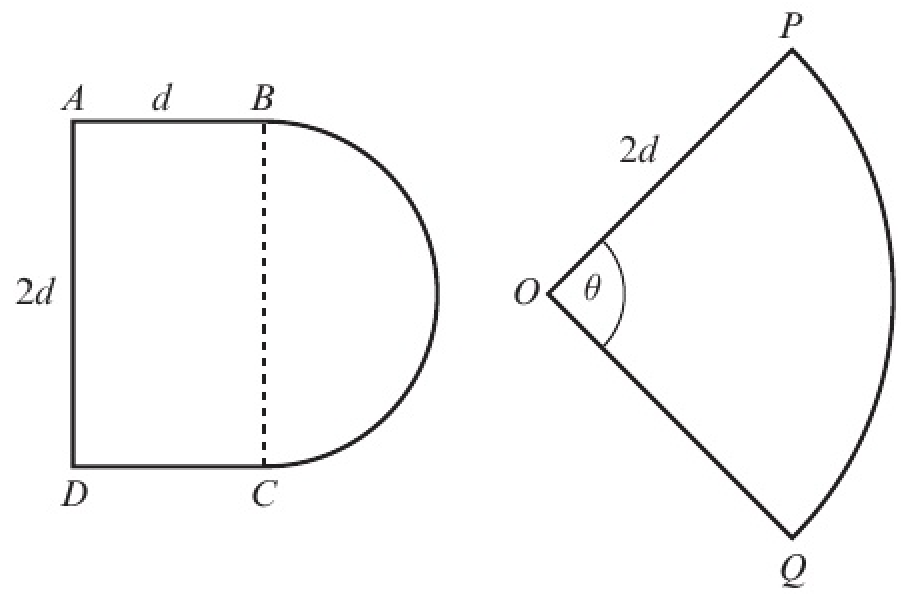
b Calculate the value of R_1. **(2 marks)**

c Calculate $R_1 : R_2$ in the form $1 : p$, giving the value of p to 3 significant figures. **(3 marks)**



(E/P) **6** The diagrams show the cross-sections of two drawer handles. Shape X is a rectangle $ABCD$ joined to a semicircle with BC as diameter. The length $AB = d$ cm and $BC = 2d$ cm. Shape Y is a sector OPQ of a circle with centre O and radius $2d$ cm. Angle POQ is θ radians.

a Given that the areas of shapes X and Y are equal, prove that $\theta = 1 + \frac{\pi}{4}$. **(5 marks)**



Shape X Shape Y

Using this value of θ, and given that $d = 3$, find in terms of π:

b the perimeter of shape X **(3 marks)**

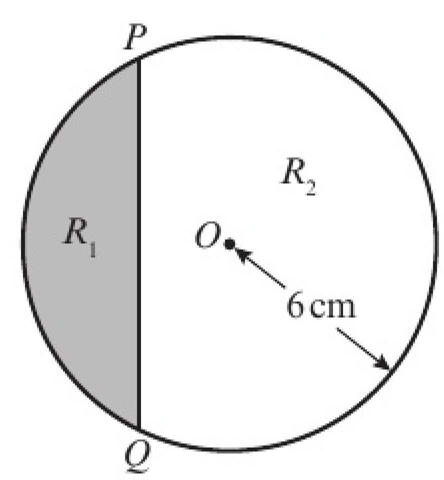
c the perimeter of shape Y. **(3 marks)**

d Hence find the difference, in mm, between the perimeters of shapes X and Y. **(1 mark)**

(E/P) **7** The diagram shows a circle with centre O and radius 6 cm. The chord PQ divides the circle into a minor segment R_1 of area A_1 cm^2 and a major segment R_2 of area A_2 cm^2. The chord PQ subtends an angle θ radians at O.

a Show that $A_1 = 18(\theta - \sin\theta)$. **(2 marks)**

b Given that $A_2 = 3A_1$, show that $\sin\theta = \theta - \frac{\pi}{2}$. **(4 marks)**



(E/P) **8** Triangle ABC has $AB = 9$ cm, $BC = 10$ cm and $CA = 5$ cm. A circle, centre A and radius 3 cm, intersects AB and AC at P and Q respectively, as shown in the diagram.

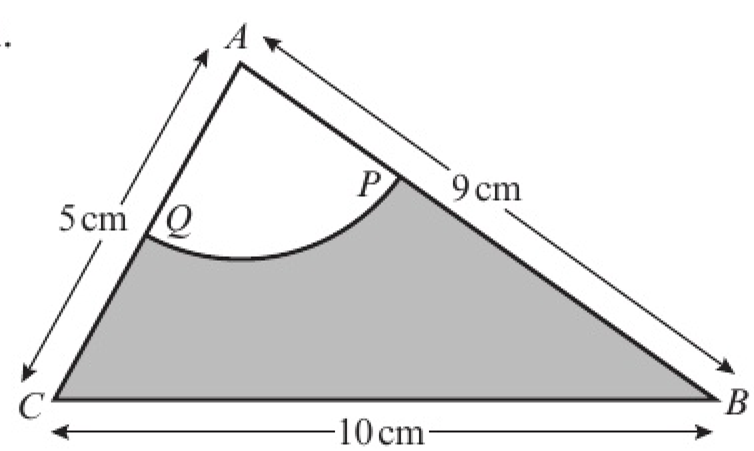
a Show that, to 3 decimal places, $\angle BAC = 1.504$ radians. **(2 marks)**

b Calculate:

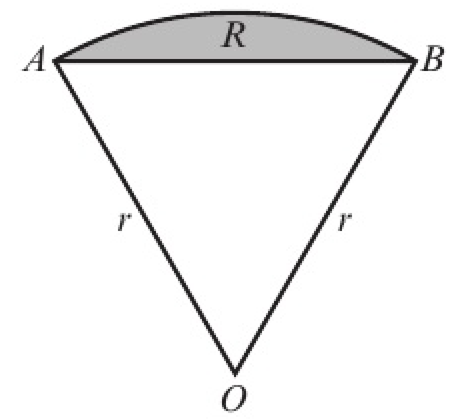
i the area, in cm^2, of the sector APQ

ii the area, in cm^2, of the shaded region $BPQC$

iii the perimeter, in cm, of the shaded region $BPQC$. **(8 marks)**



E/P **9** The diagram shows the sector OAB of a circle of radius r cm. The area of the sector is 15 cm^2 and $\angle AOB = 1.5$ radians.



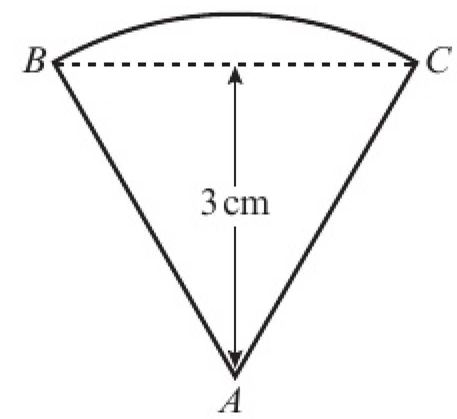
a Prove that $r = 2\sqrt{5}$. **(2 marks)**

b Find, in cm, the perimeter of the sector OAB. **(3 marks)**

The segment R, shaded in the diagram, is enclosed by the arc AB and the straight line AB.

c Calculate, to 3 decimal places, the area of R. **(2 marks)**

E/P **10** The shape of a badge is a sector ABC of a circle with centre A and radius AB, as shown in the diagram. The triangle ABC is equilateral and has perpendicular height 3 cm.

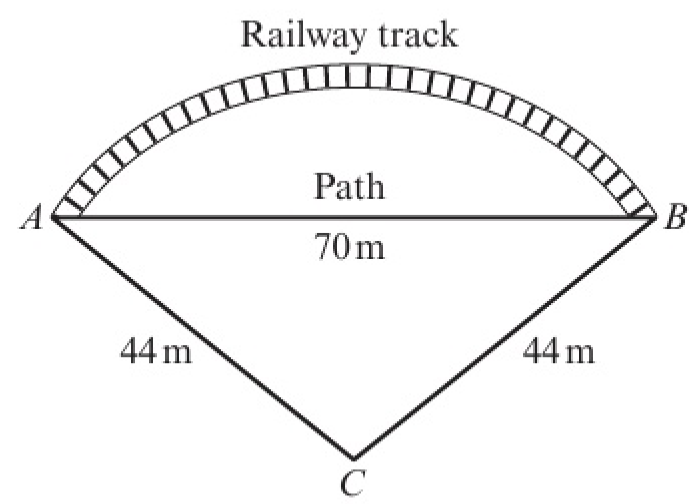


a Find, in surd form, the length of AB. **(2 marks)**

b Find, in terms of π, the area of the badge. **(2 marks)**

c Prove that the perimeter of the badge is $\frac{2\sqrt{3}}{3}(\pi + 6)$ cm. **(4 marks)**

E **11** There is a straight path of length 70 m from the point A to the point B. The points are joined also by a railway track in the form of an arc of the circle whose centre is C and whose radius is 44 m, as shown in the diagram.



a Show that the size, to 2 decimal places, of $\angle ACB$ is 1.84 radians. **(2 marks)**

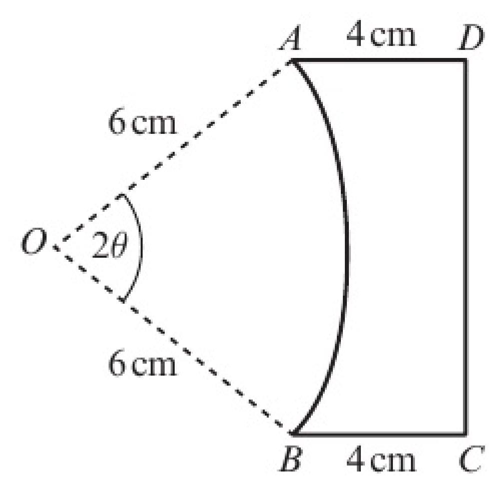
b Calculate:

i the length of the railway track

ii the shortest distance from C to the path

iii the area of the region bounded by the railway track and the path. **(6 marks)**

P **12** The diagram shows the cross-section $ABCD$ of a glass prism. $AD = BC = 4$ cm and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross-section is $2(7 + \pi)$ cm,



a show that $(2\theta + 2\sin\theta - 1) = \frac{\pi}{3}$

b verify that $\theta = \frac{\pi}{6}$

c find the area of the cross-section.

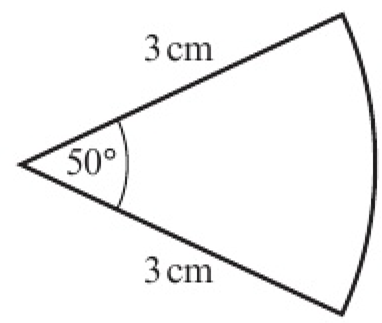
P **13** Two circles C_1 and C_2, both of radius 12 cm, have centres O_1 and O_2 respectively. O_1 lies on the circumference of C_2; O_2 lies on the circumference of C_1. The circles intersect at A and B, and enclose the region R.

a Show that $\angle AO_1B = \frac{2\pi}{3}$.

b Hence write down, in terms of π, the perimeter of R.

c Find the area of R, giving your answer to 3 significant figures.

E/P **14** A teacher asks a student to find the area of the following sector. The attempt is shown below.



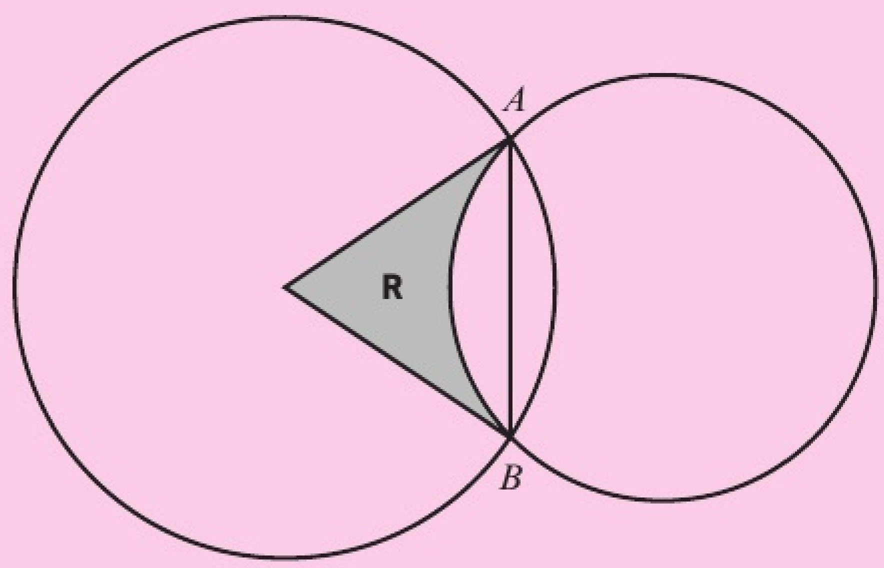
$$\text{Area} = \tfrac{1}{2}r^2\theta$$
$$= \tfrac{1}{2} \times 3^2 \times 50$$
$$= 225\text{ cm}^2$$

a Identify the mistake made by the student. **(1 mark)**

b Calculate the correct area of the sector. **(2 marks)**

Challenge

Two circles of radii 10 cm and 8 cm respectively intersect at points A and B such that the length of line AB is 14 cm.



a For each separate circle, calculate, in radians to 3 d.p., the angle subtended at the centre by the arc AB.

b Hence, or otherwise, calculate to 1 d.p. the area of the shaded region **R**, bounded by the two radii of the larger circle and the arc of the smaller circle.

Summary of key points

1 • 2π radians = 360° • π radians = 180° • 1 radian = $\frac{180°}{\pi}$

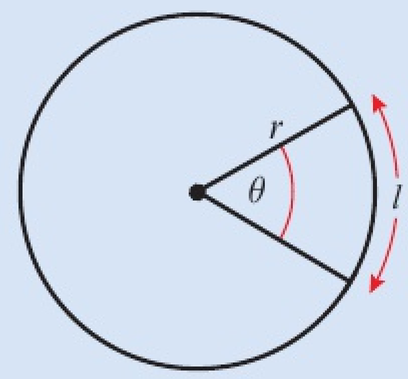
2 If the arc AB has length r then $\angle AOB$ = 1 radian (or 1 rad or 1^c)

3 • $30° = \frac{\pi}{6}$ radians • $45° = \frac{\pi}{4}$ radians • $60° = \frac{\pi}{3}$ radians

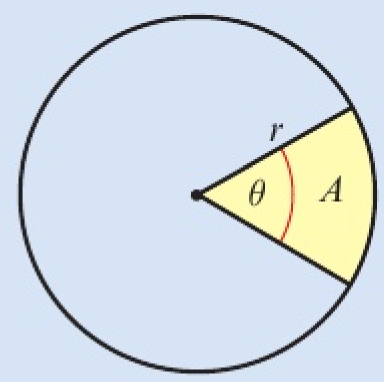
• $90° = \frac{\pi}{2}$ radians • $180° = \pi$ radians • $360° = 2\pi$ radians

4 A radian is the angle subtended at the centre of a circle by an arc whose length is equal to that of the radius of the circle.

5 To find the arc length l of a sector of a circle, use the formula $l = r\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.

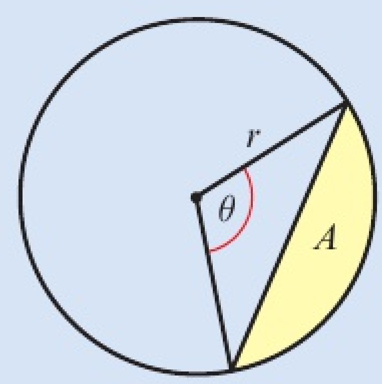


6 To find the area A of a sector of a circle, use the formula $A = \frac{1}{2}r^2\theta$, where r is the radius of the circle and θ is the angle, in radians, contained by the sector.



7 The area of a segment in a circle of radius r is

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$



8 DIFFERENTIATION

4.1 4.2 4.3

Learning objectives

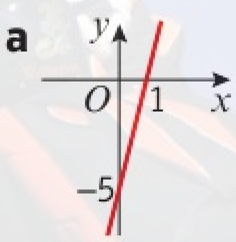
After completing this chapter you should be able to:

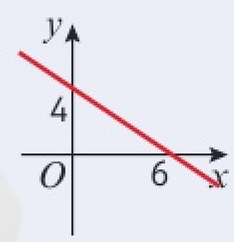
- Find the derivative, $f'(x)$ or $\frac{dy}{dx}$, of a simple function → pages 154–163
- Use the derivative to solve problems involving gradients, tangents and normals → pages 163–165
- Find the second derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$, of a simple function → pages 165–166

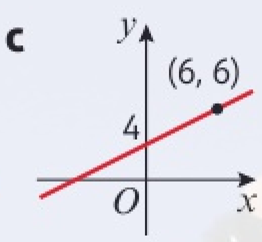
Differentiation is part of calculus, one of the most powerful tools in mathematics. You will use differentiation in mechanics to model rates of change, such as speed and acceleration.

Prior knowledge check

1 Find the gradients of these lines.

a 

b 

c 

← Section 5.1

2 Write each of these expressions in the form x^n where n is a positive or negative real number.

a $x^3 \times x^7$ **b** $\sqrt[3]{x^2}$

c $\frac{x^2 \times x^3}{x^6}$ **d** $\sqrt{\frac{x^2}{\sqrt{x}}}$ ← Sections 1.1, 1.4

3 Find the equation of the straight line that passes through:

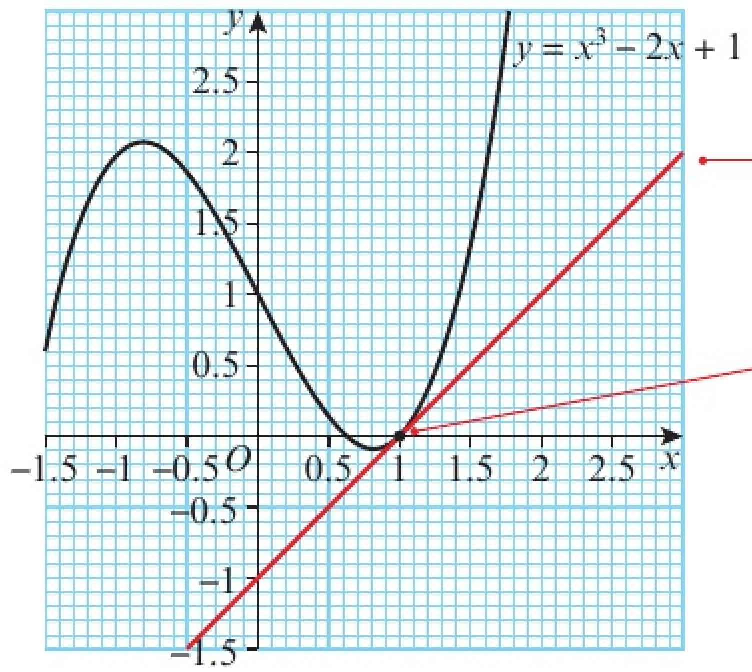
a (0, –2) and (6, 1) **b** (3, 7) and (9, 4)
c (10, 5) and (–2, 8) ← Section 5.2

4 Find the equation of the perpendicular to the line $y = 2x - 5$ at the point (2, –1). ← Section 5.3

8.1 Gradients of curves

The gradient of a curve is constantly changing. You can use a tangent to find the gradient of a curve at any point on the curve. The tangent to a curve at a point A is the straight line that just touches the curve at A.

- The **gradient** of a curve at a given point is defined as the gradient of the **tangent** to the curve at that point.

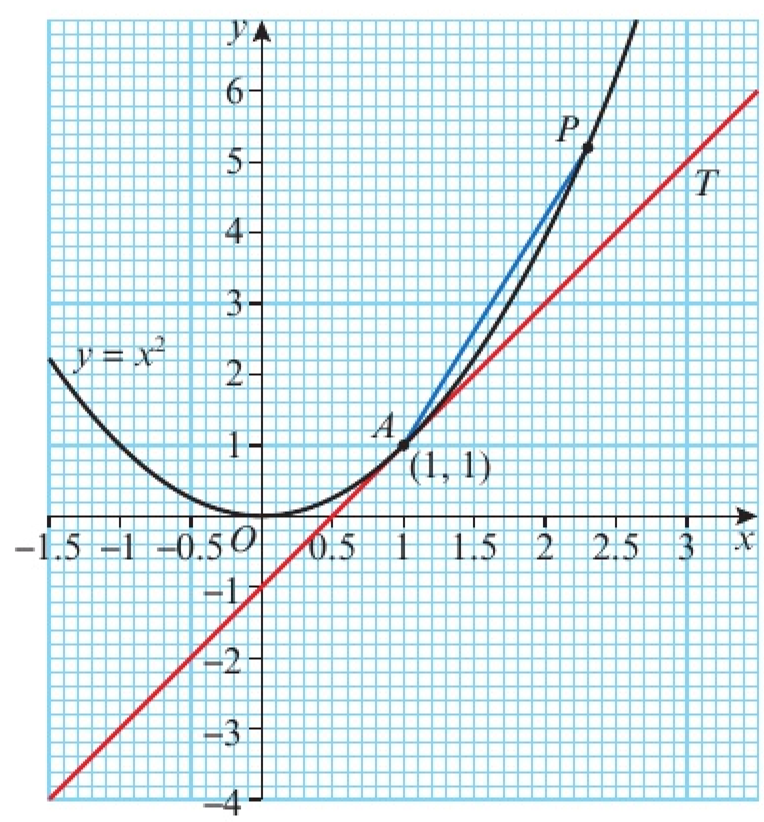


The tangent to the curve at (1, 0) has gradient 1, so the gradient of the curve at the point (1, 0) is equal to 1.

The tangent **just touches** the curve at (1, 0). It does not cut the curve at this point, although it may cut the curve at another point.

Example 1

The diagram shows the curve with equation $y = x^2$.
The tangent, T, to the curve at the point $A(1, 1)$ is shown.
Point A is joined to point P by the chord AP.



a Calculate the gradient of the tangent, T.

b Calculate the gradient of the chord AP when P has coordinates:

i (2, 4)

ii (1.5, 2.25)

iii (1.1, 1.21)

iv (1.01, 1.0201)

v $(1 + h, (1 + h)^2)$

c Comment on the relationship between your answers to parts **a** and **b**.

a Gradient of tangent $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{3 - 1}{2 - 1}$

$= 2$

Use the formula for the gradient of a straight line between points (x_1, y_1) and (x_2, y_2). ← Section 5.1

The points used are (1, 1) and (2, 3).

b **i** Gradient of chord joining (1, 1) to (2, 4)

$= \dfrac{4 - 1}{2 - 1}$

$= 3$

Online Explore the gradient of the chord AP using technology.

ii Gradient of the chord joining (1, 1) to (1.5, 2.25)

$= \dfrac{2.25 - 1}{1.5 - 1}$

$= \dfrac{1.25}{0.5}$

$= 2.5$

This time (x_1, y_1) is (1, 1) and (x_2, y_2) is (1.5, 2.25).

iii Gradient of the chord joining (1, 1) to (1.1, 1.21)

$= \dfrac{1.21 - 1}{1.1 - 1}$

$= \dfrac{0.21}{0.1}$

$= 2.1$

iv Gradient of the chord joining (1, 1) to (1.01, 1.0201)

$= \dfrac{1.0201 - 1}{1.01 - 1}$

$= \dfrac{0.0201}{0.01}$

$= 2.01$

This point is closer to (1, 1) than (1.1, 1.21) is.

This gradient is closer to 2.

v Gradient of the chord joining (1, 1) to $(1 + h, (1 + h)^2)$

$= \dfrac{(1 + h)^2 - 1}{(1 + h) - 1}$

$= \dfrac{1 + 2h + h^2 - 1}{1 + h - 1}$

$= \dfrac{2h + h^2}{h}$

$= 2 + h$

h is a constant.

$(1 + h)^2 = (1 + h)(1 + h) = 1 + 2h + h^2$

This becomes $\dfrac{h(2 + h)}{h}$

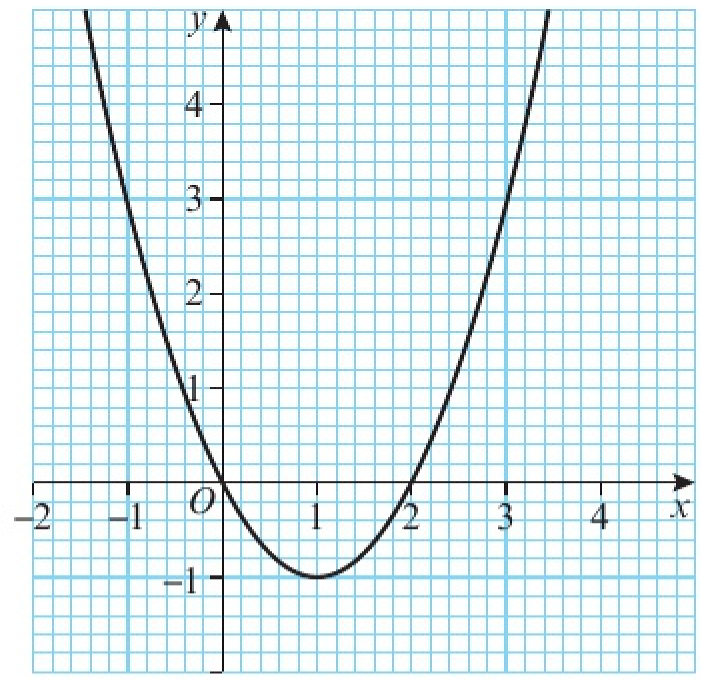
You can use this formula to confirm the answers to questions **i** to **iv**. For example, when $h = 0.5$, $(1 + h, (1 + h)^2) = (1.5, 2.25)$ and the gradient of the chord is $2 + 0.5 = 2.5$.

c As P gets closer to A, the gradient of the chord AP gets closer to the gradient of the tangent at A.

As h gets closer to zero, $2 + h$ gets closer to 2, so the gradient of the chord gets closer to the gradient of the tangent.

Exercise 8A SKILLS REASONING/ARGUMENTATION

1 The diagram shows the curve with equation $y = x^2 - 2x$.



a Copy and complete this table showing estimates for the gradient of the curve.

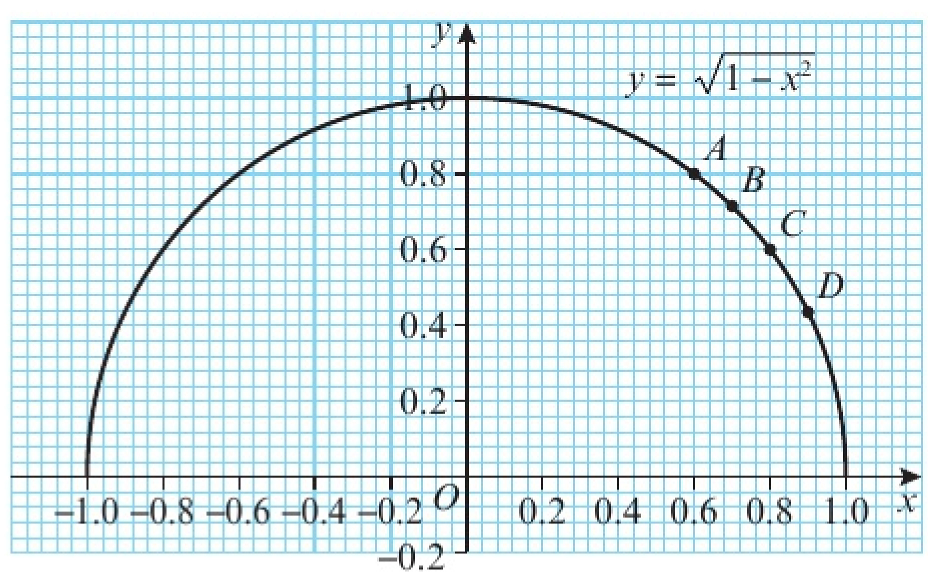
x-coordinate	−1	0	1	2	3
Estimate for gradient of curve					

b Write a hypothesis about the gradient of the curve at the point where $x = p$.

c Test your hypothesis by estimating the gradient of the graph at the point (1.5, −0.75).

Hint Place a ruler on the graph to approximate each tangent.

2 The diagram shows the curve with equation $y = \sqrt{1 - x^2}$.
The point A has coordinates (0.6, 0.8).
The points B, C and D lie on the curve with x-coordinates 0.7, 0.8 and 0.9 respectively.



a Verify that point A lies on the curve.

b Use a ruler to estimate the gradient of the curve at point A.

c Find the gradient of the line segments:

i AD

ii AC

iii AB

Hint Use algebra for part **c**.

d Comment on the relationship between your answers to parts **b** and **c**.

3 F is the point with coordinates (3, 9) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point F to the points with coordinates:

i (4, 16) **ii** (3.5, 12.25) **iii** (3.1, 9.61)

iv (3.01, 9.0601) **v** $(3 + h, (3 + h)^2)$

b What do you deduce about the gradient of the tangent at the point (3, 9)?

4 G is the point with coordinates (4, 16) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point G to the points with coordinates:

i (5, 25) **ii** (4.5, 20.25) **iii** (4.1, 16.81)

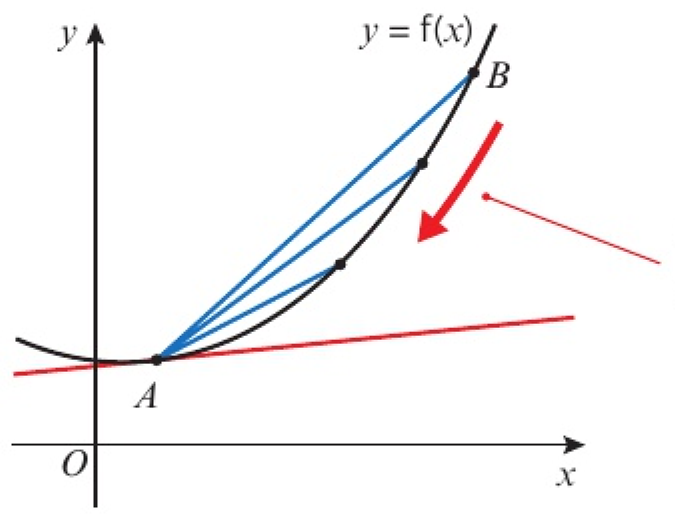
iv (4.01, 16.0801) **v** $(4 + h, (4 + h)^2)$

b What do you deduce about the gradient of the tangent at the point (4, 16)?

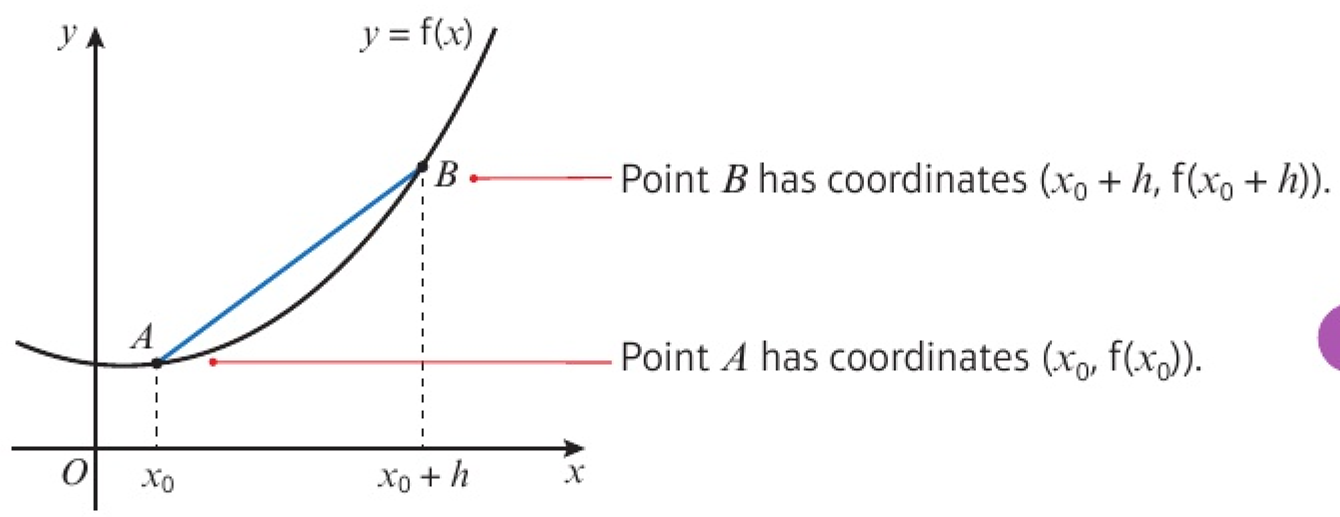
8.2 Finding the derivative

Differentiation from first principles is not examined. It is included here to give a detailed account of where differentiation originates.

You can use algebra to find the exact gradient of a curve at a given point. This diagram shows two points, A and B, that lie on the curve with equation $y = f(x)$.



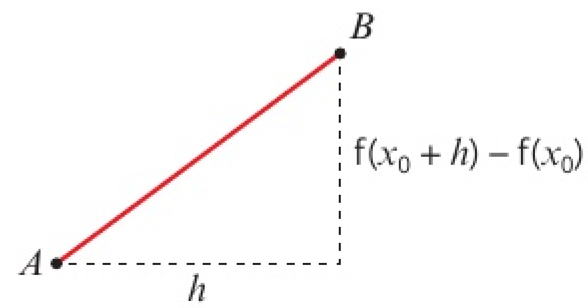
You can **formalise** this approach by letting the x-coordinate of A be x_0 and the x-coordinate of B be $x_0 + h$. Consider what happens to the gradient of AB as h gets smaller.



Notation h represents a **small change** in the value of x. You can also use δx to represent this small change. It is pronounced 'delta x'.

The vertical distance from A to B is $f(x_0 + h) - f(x_0)$.
The horizontal distance is $x_0 + h - x_0 = h$.

So the gradient of AB is $\frac{f(x_0 + h) - f(x_0)}{h}$.



As h gets smaller, the gradient of AB gets closer to the gradient of the tangent to the curve at A. This means that the gradient of the **curve** at A is the **limit** of this expression as the value of h tends to 0.

You can use this to define the **gradient function**.

- The gradient function, or **derivative**, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

  $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

  The gradient function can be used to find the gradient of the curve for any value of x.

Notation $\lim_{h \to 0}$ means 'the limit as h tends to 0'. You can't evaluate the expression when $h = 0$, but as h gets smaller the expression gets closer to a fixed (or **limiting**) value.

Using this rule to find the derivative is called **differentiating from first principles**.

Example 2 SKILLS ANALYSIS

The point A with coordinates (4, 16) lies on the curve with equation $y = x^2$.
At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0} (8 + h)$.

b Deduce the value of g.

a $g = \lim_{h \to 0} \frac{f(4 + h) - f(4)}{h}$

Use the definition of the derivative with $x = 4$.

$= \lim_{h \to 0} \frac{(4 + h)^2 - 4^2}{h}$

The function is $f(x) = x^2$. Remember to square everything inside the brackets.

$= \lim_{h \to 0} \frac{16 + 8h + h^2 - 16}{h}$

$= \lim_{h \to 0} \frac{8h + h^2}{h}$

The 16 and the −16 cancel, and you can cancel h in the fraction.

$= \lim_{h \to 0} (8 + h)$

b $g = 8$

As $h \to 0$ the limiting value is 8, so the gradient at point A is 8.

Example 3

Prove, from first principles, that the derivative of x^3 is $3x^2$.

$\mathrm{f}(x) = x^3$

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x)^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

As $h \to 0$, $3xh \to 0$ and $h^2 \to 0$.

So $\mathrm{f}'(x) = 3x^2$

Hint 'From first principles' means that you have to use the definition of the derivative. You are starting your proof with a known definition, so this is an example of a proof by deduction.

$(x + h)^3 = (x + h)(x + h)^2$
$= (x + h)(x^2 + 2hx + h^2)$
which expands to give $x^3 + 3x^2h + 3xh^2 + h^3$

Factorise the numerator.

Any terms containing h, h^2, h^3, etc will have a limiting value of 0 as $h \to 0$.

Exercise 8B

SKILLS REASONING/ARGUMENTATION

1 For the function $\mathrm{f}(x) = x^2$, use the definition of the derivative to show that:

a $\mathrm{f}'(2) = 4$ **b** $\mathrm{f}'(-3) = -6$ **c** $\mathrm{f}'(0) = 0$ **d** $\mathrm{f}'(50) = 100$

2 $\mathrm{f}(x) = x^2$

a Show that $\mathrm{f}'(x) = \lim_{h \to 0} (2x + h)$. **b** Hence deduce that $\mathrm{f}'(x) = 2x$.

3 The point A with coordinates $(-2, -8)$ lies on the curve with equation $y = x^3$. At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0} (12 - 6h + h^2)$. **b** Deduce the value of g.

(P) **4** The point A with coordinates $(-1, 4)$ lies on the curve with equation $y = x^3 - 5x$.
The point B also lies on the curve and has x-coordinate $(-1 + h)$.

a Show that the gradient of the line segment AB is given by $h^2 - 3h - 2$.

b Deduce the gradient of the curve at point A.

Problem-solving Draw a sketch showing points A and B and the chord between them.

(E/P) **5** Prove, from first principles, that the derivative of $6x$ is 6. **(3 marks)**

(E/P) **6** Prove, from first principles, that the derivative of $4x^2$ is $8x$. **(4 marks)**

(E/P) **7** $\mathrm{f}(x) = ax^2$, where a is a constant. Prove, from first principles, that $\mathrm{f}'(x) = 2ax$. **(4 marks)**

Challenge

$f(x) = \frac{1}{x}$

a Given that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, show that $f'(x) = \lim_{h \to 0} \frac{-1}{x^2 + xh}$.

b Deduce that $f'(x) = -\frac{1}{x^2}$.

8.3 Differentiating x^n

You can use the definition of the derivative to find an expression for the derivative of x^n where n is any number. This is called **differentiation**.

- For all real values of n, and for a constant a:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$
    If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
  - If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$
    If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

Notation

$f'(x)$ and $\frac{dy}{dx}$ both represent the derivative. You usually use $\frac{dy}{dx}$ when an expression is given in the form $y = \ldots$

Example 4

Find the derivative, $f'(x)$, when $f(x)$ equals:

a x^6 **b** $x^{\frac{1}{2}}$ **c** x^{-2} **d** $x^2 \times x^3$ **e** $\frac{x}{x^5}$

a $f(x) = x^6$
So $f'(x) = 6x^5$

Multiply by the power, then subtract 1 from the power:
$6 \times x^{6-1} = 6x^5$

b $f(x) = x^{\frac{1}{2}}$
So $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$= \frac{1}{2\sqrt{x}}$

The new power is $\frac{1}{2} - 1 = -\frac{1}{2}$

$x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

← Section 1.4

c $f(x) = x^{-2}$
So $f'(x) = -2x^{-3}$
$= -\frac{2}{x^3}$

You can leave your answer in this form or write it as a fraction.

d $f(x) = x^2 \times x^3$
$= x^5$
So $f'(x) = 5x^4$

You need to write the function in the form x^n before you can use the rule.

$x^2 \times x^3 = x^{2+3}$
$= x^5$

e $f(x) = x \div x^5$

$= x^{-4}$

So $f'(x) = -4x^{-5}$

$= -\frac{4}{x^5}$

Use the laws of indices to simplify the fraction: $x^1 \div x^5 = x^{1-5} = x^{-4}$

Example 5

Find $\frac{dy}{dx}$ when y equals:

a $7x^3$ **b** $-4x^{\frac{1}{2}}$ **c** $3x^{-2}$ **d** $\frac{8x^7}{3x}$ **e** $\sqrt{36x^3}$

a $\frac{dy}{dx} = 7 \times 3x^{3-1} = 21x^2$

Use the rule for differentiating ax^n with $a = 7$ and $n = 3$. Multiply by 3 then subtract 1 from the power.

b $\frac{dy}{dx} = -4 \times \frac{1}{2}x^{-\frac{1}{2}} = -2x^{-\frac{1}{2}} = -\frac{2}{\sqrt{x}}$

This is the same as differentiating $x^{\frac{1}{2}}$ then multiplying the result by −4.

c $\frac{dy}{dx} = 3 \times -2x^{-3} = -6x^{-3} = -\frac{6}{x^3}$

d $y = \frac{8}{3}x^6$

Write the expression in the form ax^n. Remember a can be any number, including fractions.

$\frac{dy}{dx} = 6 \times \frac{8}{3}x^5 = 16x^5$

e $y = \sqrt{36} \times \sqrt{x^3} = 6 \times (x^3)^{\frac{1}{2}} = 6x^{\frac{3}{2}}$

$\frac{dy}{dx} = 6 \times \frac{3}{2}x^{\frac{1}{2}} = 9x^{\frac{1}{2}} = 9\sqrt{x}$

$\frac{3}{2} - 1 = \frac{1}{2}$

Simplify the number part as much as possible.

Exercise 8C SKILLS INTERPRETATION

Hint Make sure that the functions are in the form x^n before you differentiate.

1 Find $f'(x)$ given that $f(x)$ equals:

a x^7 **b** x^8 **c** x^4 **d** $x^{\frac{1}{3}}$ **e** $x^{\frac{1}{4}}$ **f** $\sqrt[3]{x}$

g x^{-3} **h** x^{-4} **i** $\frac{1}{x^2}$ **j** $\frac{1}{x^5}$ **k** $\frac{1}{\sqrt{x}}$ **l** $\frac{1}{\sqrt[3]{x}}$

m $x^3 \times x^6$ **n** $x^2 \times x^3$ **o** $x \times x^2$ **p** $\frac{x^2}{x^4}$ **q** $\frac{x^3}{x^2}$ **r** $\frac{x^6}{x^3}$

2 Find $\frac{dy}{dx}$ given that y equals:

a $3x^2$ **b** $6x^9$ **c** $\frac{1}{2}x^4$ **d** $20x^{\frac{1}{4}}$ **e** $6x^{\frac{5}{4}}$

f $10x^{-1}$ **g** $\frac{4x^6}{2x^3}$ **h** $\frac{x}{8x^5}$ **i** $-\frac{2}{\sqrt{x}}$ **j** $\sqrt{\frac{5x^4 \times 10x}{2x^2}}$

3 Find the gradient of the curve with equation $y = 3\sqrt{x}$ at the point where:

a $x = 4$ b $x = 9$

c $x = \frac{1}{4}$ d $x = \frac{9}{16}$

E/P 4 Given that $2y^2 - x^3 = 0$ and $y > 0$, find $\frac{dy}{dx}$. **(2 marks)**

Problem-solving
Try rearranging unfamiliar equations into a form you recognise.

8.4 Differentiating quadratics

You can differentiate a function with more than one term by differentiating the terms **one-at-a-time**. The highest power of x in a **quadratic function** is x^2, so the highest power of x in its derivative will be x.

■ **For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by**

$$\frac{dy}{dx} = 2ax + b$$

Links The derivative is a **straight line** with gradient $2a$. It crosses the x-axis once, at the point where the quadratic curve has zero gradient. This is the **turning point** of the quadratic curve. ← Section 5.1

You can find this expression for $\frac{dy}{dx}$ by differentiating each of the terms one-at-a-time:

ax^2 → Differentiate → $2ax^1 = 2ax$
The quadratic term tells you the slope of the gradient function.

$bx = bx^1$ → Differentiate → $1bx^0 = b$
An x term differentiates to give a constant.

c → Differentiate → 0
Constant terms disappear when you differentiate.

Example 6 SKILLS INTERPRETATION

Find $\frac{dy}{dx}$ given that y equals:

a $x^2 + 3x$ b $8x^2 - 7$ c $4x^2 - 3x + 5$

a $y = x^2 + 3x$

So $\frac{dy}{dx} = 2x + 3$

Differentiate the terms one-at-a-time.

b $y = 8x^2 - 7$

The constant term disappears when you differentiate. The line $y = -7$ has **zero gradient**.

So $\frac{dy}{dx} = 16x$

c $y = 4x^2 - 3x + 5$

$4x^2 - 3x + 5$ is a quadratic expression with $a = 4$, $b = -3$ and $c = 5$.

So $\frac{dy}{dx} = 8x - 3$

The derivative is $2ax + b = 2 \times 4x - 3 = 8x - 3$.

Example 7

Let $f(x) = 4x^2 - 8x + 3$.

a Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$.

b Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

a As $y = 4x^2 - 8x + 3$

$\frac{dy}{dx} = f'(x) = 8x - 8 + 0$

So $f'\left(\frac{1}{2}\right) = -4$

Differentiate to find the gradient function. Then substitute the x-coordinate value to obtain the gradient.

b $\frac{dy}{dx} = f'(x) = 8x - 8 = 8$

So $x = 2$

So $y = f(2) = 3$

The point where the gradient is 8 is (2, 3).

Put the gradient function equal to 8. Then solve the equation you have obtained to give the value of x.

Substitute this value of x into $f(x)$ to give the value of y and interpret your answer in words.

c $4x^2 - 8x + 3 = 4x - 5$

$4x^2 - 12x + 8 = 0$

$x^2 - 3x + 2 = 0$

$(x - 2)(x - 1) = 0$

So $x = 1$ or $x = 2$

At $x = 1$, the gradient is 0.

At $x = 2$, the gradient is 8, as in part **b**.

To find the points of intersection, set the equation of the curve equal to the equation of the line. Solve the resulting quadratic equation to find the x-coordinates of the points of intersection. ← **Section 2.1**

Substitute the values of x into $f'(x) = 8x - 8$ to give the gradients at the specified points.

Online Use your calculator to check solutions to quadratic equations quickly.

Exercise 8D SKILLS ANALYSIS

1 Find $\frac{dy}{dx}$ when y equals:

a $2x^2 - 6x + 3$ **b** $\frac{1}{2}x^2 + 12x$ **c** $4x^2 - 6$

d $8x^2 + 7x + 12$ **e** $5 + 4x - 5x^2$

2 Find the gradient of the curve with equation:

a $y = 3x^2$, at the point (2, 12) **b** $y = x^2 + 4x$, at the point (1, 5)

c $y = 2x^2 - x - 1$, at the point (2, 5) **d** $y = \frac{1}{2}x^2 + \frac{3}{2}x$, at the point (1, 2)

e $y = 3 - x^2$, at the point (1, 2) **f** $y = 4 - 2x^2$, at the point (−1, 2)

3 Find the y-coordinate and the value of the gradient at the point P with x-coordinate 1 on the curve with equation $y = 3 + 2x - x^2$.

4 Find the coordinates of the point on the curve with equation $y = x^2 + 5x - 4$ where the gradient is 3.

(P) **5** Find the gradients of the curve $y = x^2 - 5x + 10$ at the points A and B where the curve meets the line $y = 4$.

(P) **6** Find the gradients of the curve $y = 2x^2$ at the points C and D where the curve meets the line $y = x + 3$.

(P) **7** $f(x) = x^2 - 2x - 8$

a Sketch the graph of $y = f(x)$.

b On the same set of axes, sketch the graph of $y = f'(x)$.

c Explain why the x-coordinate of the turning point of $y = f(x)$ is the same as the x-coordinate of the point where the graph of $y = f'(x)$ crosses the x-axis.

8.5 Differentiating functions with two or more terms

You can use the rule for differentiating ax^n to differentiate functions with two or more terms. You need to be able to rearrange **each term** into the form ax^n, where a is a constant and n is a real number. Then you can differentiate the terms one-at-a-time.

- If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$

Example 8

Find $\frac{dy}{dx}$ when y equals:

a $4x^3 + 2x$ **b** $x^3 + x^2 - x^{\frac{1}{2}}$ **c** $\frac{1}{3}x^{\frac{1}{2}} + 4x^2$

a $y = 4x^3 + 2x$

So $\frac{dy}{dx} = 12x^2 + 2$

Differentiate the terms one-at-a-time.

b $y = x^3 + x^2 - x^{\frac{1}{2}}$

So $\frac{dy}{dx} = 3x^2 + 2x - \frac{1}{2}x^{-\frac{1}{2}}$

Be careful with the third term. You multiply the term by $\frac{1}{2}$ and then reduce the power by 1 to get $-\frac{1}{2}$

c $y = \frac{1}{3}x^{\frac{1}{2}} + 4x^2$

Check that each term is in the form ax^n before differentiating.

So $\frac{dy}{dx} = \frac{1}{3} \times \frac{1}{2}x^{-\frac{1}{2}} + 8x$

$= \frac{1}{6}x^{-\frac{1}{2}} + 8x$

Example 9

Differentiate:

a $\frac{1}{4\sqrt{x}}$ **b** $x^3(3x + 1)$ **c** $\frac{x - 2}{x^2}$

a Let $y = \frac{1}{4\sqrt{x}}$

$= \frac{1}{4}x^{-\frac{1}{2}}$

Therefore $\frac{dy}{dx} = -\frac{1}{8}x^{-\frac{3}{2}}$

Use the laws of indices to write the expression in the form ax^n.

$$\frac{1}{4\sqrt{x}} = \frac{1}{4} \times \frac{1}{\sqrt{x}} = \frac{1}{4} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{4}x^{-\frac{1}{2}}$$

b Let $y = x^3(3x + 1)$

$= 3x^4 + x^3$

Therefore $\frac{dy}{dx} = 12x^3 + 3x^2$

$= 3x^2(4x + 1)$

Multiply out the brackets to give a **polynomial** function.

Differentiate each term.

c Let $y = \frac{x - 2}{x^2}$

$= \frac{1}{x} - \frac{2}{x^2}$

$= x^{-1} - 2x^{-2}$

Therefore $\frac{dy}{dx} = -x^{-2} + 4x^{-3}$

$= -\frac{1}{x^2} + \frac{4}{x^3}$

$= \frac{4 - x}{x^3}$

Express the single fraction as two separate fractions, and simplify: $\frac{x}{x^2} = \frac{1}{x}$

Write each term in the form ax^n then differentiate.

You can write the answer as a single fraction with denominator x^3.

Exercise 8E SKILLS INTERPRETATION

1 Differentiate:

a $x^4 + x^{-1}$ **b** $2x^5 + 3x^{-2}$ **c** $6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + 4$

2 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x^3 - 3x + 2$, and A is at $(-1, 4)$ **b** $f(x) = 3x^2 + 2x^{-1}$, and A is at $(2, 13)$

3 Find the point(s) on the curve with equation $y = f(x)$, where the gradient is zero:

a $f(x) = x^2 - 5x$ **b** $f(x) = x^3 - 9x^2 + 24x - 20$

c $f(x) = x^{\frac{3}{2}} - 6x + 1$ **d** $f(x) = x^{-1} + 4x$

4 Differentiate:

a $2\sqrt{x}$ **b** $\frac{3}{x^2}$ **c** $\frac{1}{3x^3}$ **d** $\frac{1}{3}x^3(x - 2)$

e $\frac{2}{x^3} + \sqrt{x}$ **f** $\sqrt[3]{x} + \frac{1}{2x}$ **g** $\frac{2x + 3}{x}$ **h** $\frac{3x^2 - 6}{x}$

i $\frac{2x^3 + 3x}{\sqrt{x}}$ **j** $x(x^2 - x + 2)$ **k** $3x^2(x^2 + 2x)$ **l** $(3x - 2)\left(4x + \frac{1}{x}\right)$

5 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x(x + 1)$, and A is at $(0, 0)$

b $f(x) = \frac{2x - 6}{x^2}$, and A is at $(3, 0)$

c $f(x) = \frac{1}{\sqrt{x}}$, and A is at $\left(\frac{1}{4}, 2\right)$

d $f(x) = 3x - \frac{4}{x^2}$, and A is at $(2, 5)$

E/P **6** $f(x) = \frac{12}{p\sqrt{x}} + x$, where p is a real constant and $x > 0$.

Given that $f'(2) = 3$, find p, giving your answer in the form $a\sqrt{2}$ where a is a rational number. **(4 marks)**

8.6 Gradients, tangents and normals

You can use the derivative to find the equation of the tangent to a curve at a given point. On the curve with equation $y = f(x)$, the gradient of the tangent at a point A with x-coordinate a will be $f'(a)$.

- The tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

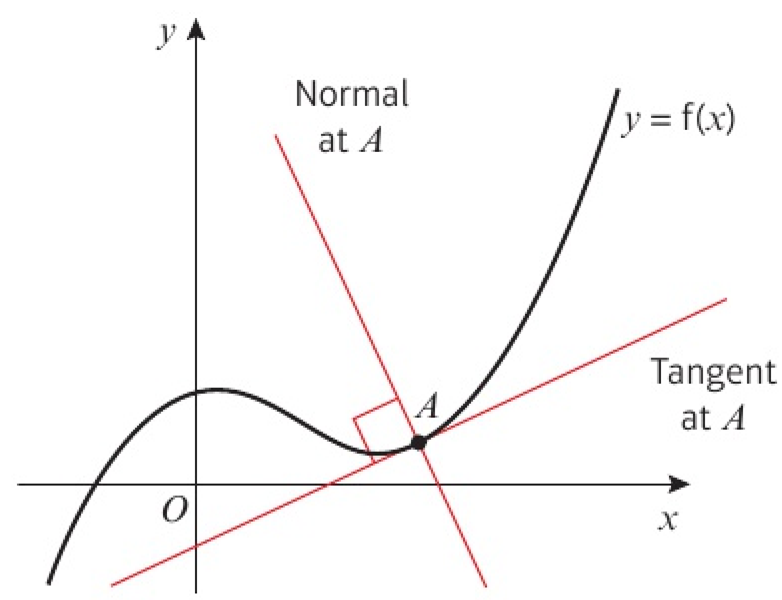
$$y - f(a) = f'(a)(x - a)$$

Links The equation of a straight line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$. ← Section 5.2

The **normal** to a curve at point A is the straight line through A which is perpendicular to the tangent to the curve at A. The gradient of the normal will be $-\frac{1}{f'(a)}$.

- The normal to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$



Example 10

Find the equation of the tangent to the curve $y = x^3 - 3x^2 + 2x - 1$ at the point (3, 5).

$y = x^3 - 3x^2 + 2x - 1$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

When $x = 3$, the gradient is 11.

So the equation of the tangent at (3, 5) is

$y - 5 = 11(x - 3)$

$y = 11x - 28$

First differentiate to determine the gradient function.

Then substitute for x to calculate the value of the gradient of the curve and of the tangent when $x = 3$.

You can now use the line equation and simplify.

Example 11

Find the equation of the normal to the curve with equation $y = 8 - 3\sqrt{x}$ at the point where $x = 4$.

$y = 8 - 3\sqrt{x}$

$= 8 - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{1}{2}}$

When $x = 4$, $y = 2$ and gradient of curve and of tangent $= -\frac{3}{4}$

So gradient of normal is $\frac{4}{3}$.

Equation of normal is

$y - 2 = \frac{4}{3}(x - 4)$

$3y - 6 = 4x - 16$

$3y - 4x + 10 = 0$

Write each term in the form ax^n and differentiate to obtain the gradient function, which you can use to find the gradient at any point.

Find the y-coordinate when $x = 4$ by substituting into the equation of the curve and calculating $8 - 3\sqrt{4} = 8 - 6 = 2$.

Find the gradient of the curve, by calculating

$$\frac{dy}{dx} = -\frac{3}{2}(4)^{-\frac{1}{2}} = -\frac{3}{2} \times \frac{1}{2} = -\frac{3}{4}$$

$$\text{Gradient of normal} = -\frac{1}{\text{gradient of tangent}}$$

$$= -\frac{1}{\left(-\frac{3}{4}\right)} = \frac{4}{3}$$

Simplify by multiplying both sides by 3 and collecting terms.

Online Explore the tangent and normal to the curve using technology.

Exercise 8F

SKILLS PROBLEM SOLVING

1 Find the equation of the tangent to the curve:

a $y = x^2 - 7x + 10$, at the point (2, 0)

b $y = x + \frac{1}{x}$, at the point $\left(2, 2\frac{1}{2}\right)$

c $y = 4\sqrt{x}$, at the point (9, 12)

d $y = \frac{2x - 1}{x}$, at the point (1, 1)

e $y = 2x^3 + 6x + 10$, at the point (−1, 2)

f $y = x^2 - \frac{7}{x^2}$, at the point (1, −6)

2 Find the equation of the normal to the curve:

a $y = x^2 - 5x$, at the point (6, 6)

b $y = x^2 - \frac{8}{\sqrt{x}}$, at the point (4, 12)

(P) **3** Find the coordinates of the point where the tangent to the curve $y = x^2 + 1$ at the point (2, 5) meets the normal to the same curve at the point (1, 2).

(P) **4** Find the equations of the normals to the curve $y = x + x^3$ at the points (0, 0) and (1, 2), and find the coordinates of the point where these normals meet.

(P) **5** For $f(x) = 12 - 4x + 2x^2$, find the equations of the tangent and the normal at the point where $x = -1$ on the curve with equation $y = f(x)$.

(E/P) **6** The point P with x-coordinate $\frac{1}{2}$ lies on the curve with equation $y = 2x^2$.
The normal to the curve at P intersects the curve at points P and Q.
Find the coordinates of Q. **(6 marks)**

Problem-solving

Draw a sketch showing the curve, the point P and the normal. This will help you check that your answer makes sense.

Hint Use the discriminant to find the value of m when the line just touches the curve. ← Section 2.5

Challenge

The line L is a tangent to the curve with equation $y = 4x^2 + 1$. The line L cuts the y-axis at (0, −8) and has a positive gradient. Find the equation of L in the form $y = mx + c$.

8.7 Second order derivatives

You can find the rate of change of the gradient function by differentiating a function twice.

$y = 5x^3$ → Differentiate → $\frac{dy}{dx} = 15x^2$ → Differentiate → $\frac{d^2y}{dx^2} = 30x$

This is the gradient function. It describes the rate of change of the function with respect to x.

This is the **rate of change of the gradient function**. It is called the second order derivative. It can also be written as $f''(x)$.

- **Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$**

Notation The derivative is also called the **first order derivative** or **first derivative**. The **second order derivative** is sometimes just called the **second derivative**.

Example 12

Given that $y = 3x^5 + \frac{4}{x^2}$, find:

a $\frac{dy}{dx}$ **b** $\frac{d^2y}{dx^2}$

a $y = 3x^5 + \frac{4}{x^2}$

$= 3x^5 + 4x^{-2}$

So $\frac{dy}{dx} = 15x^4 - 8x^{-3}$

$= 15x^4 - \frac{8}{x^3}$

b $\frac{d^2y}{dx^2} = 60x^3 + 24x^{-4}$

$= 60x^3 + \frac{24}{x^4}$

Express the fraction as a negative power of x.

Differentiate once to get the first order derivative.

Differentiate a second time to get the second order derivative.

Example 13 SKILLS PROBLEM SOLVING

Given that $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$, find:

a $f'(x)$ **b** $f''(x)$

a $f(x) = 3\sqrt{x} + \frac{1}{2\sqrt{x}}$

$= 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$

$f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$

b $f''(x) = -\frac{3}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$

Don't rewrite your expression for $f'(x)$ as a fraction. It will be easier to differentiate again if you leave it in this form.

The coefficient for the second term is

$$\left(-\frac{3}{2}\right) \times \left(-\frac{1}{4}\right) = +\frac{3}{8}$$

The new power is $-\frac{3}{2} - 1 = -\frac{5}{2}$

Exercise 8G SKILLS PROBLEM SOLVING

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when y equals:

a $12x^2 + 3x + 8$ **b** $15x + 6 + \frac{3}{x}$ **c** $9\sqrt{x} - \frac{3}{x^2}$ **d** $(5x + 4)(3x - 2)$ **e** $\frac{3x + 8}{x^2}$

2 The **displacement** of a particle in metres at time t seconds is modelled by the function

$f(t) = \frac{t^2 + 2}{\sqrt{t}}$.

The acceleration of the particle in ms^{-2} is the second derivative of this function.
Find an expression for the acceleration of the particle at time t seconds.

(P) **3** Given that $y = (2x - 3)^3$, find the value of x when $\frac{d^2y}{dx^2} = 0$.

(P) **4** $f(x) = px^3 - 3px^2 + x^2 - 4$
When $x = 2$, $f''(x) = -1$. Find the value of p.

Problem-solving

When you differentiate with respect to x, you treat any other letters as constants.

Chapter review 8

SKILLS EXECUTIVE FUNCTION

E/P **1** Prove, from first principles, that the derivative of $10x^2$ is $20x$. **(4 marks)**

P **2** The point A with coordinates (1, 4) lies on the curve with equation $y = x^3 + 3x$. The point B also lies on the curve and has x-coordinate $(1 + \delta x)$.

a Show that the gradient of the line segment AB is given by $(\delta x)^2 + 3\delta x + 6$.

b Deduce the gradient of the curve at point A.

3 A curve is given by the equation $y = 3x^2 + 3 + \frac{1}{x^2}$, where $x > 0$. At the points A, B and C on the curve, $x = 1$, 2 and 3 respectively. Find the gradient of the curve at A, B and C.

E **4** Calculate the x-coordinates of the points on the curve with equation $y = 7x^2 - x^3$ at which the gradient is equal to 16. **(4 marks)**

5 Find the x-coordinates of the two points on the curve with equation $y = x^3 - 11x + 1$ where the gradient is 1. Find the corresponding y-coordinates.

E **6** The function f is defined by $f(x) = x + \frac{9}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

a Find $f'(x)$. **(2 marks)**

b Solve $f'(x) = 0$. **(2 marks)**

E **7** Given that $y = 3\sqrt{x} - \frac{4}{\sqrt{x}}$, $x > 0$, find $\frac{dy}{dx}$. **(3 marks)**

E/P **8** A curve has equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$.

a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b Find the coordinates of the point on the curve where the gradient is zero. **(2 marks)**

E **9** **a** Expand $(x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$. **(2 marks)**

b A curve has equation $y = (x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$, $x > 0$. Find $\frac{dy}{dx}$. **(2 marks)**

c Use your answer to part **b** to calculate the gradient of the curve at the point where $x = 4$. **(1 mark)**

E **10** Differentiate with respect to x:

$$2x^3 + \sqrt{x} + \frac{x^2 + 2x}{x^2}$$ **(3 marks)**

E/P **11** The curve with equation $y = ax^2 + bx + c$ passes through the point (1, 2). The gradient of the curve is zero at the point (2, 1). Find the values of a, b and c. **(5 marks)**

E/P **12** A curve C has equation $y = x^3 - 5x^2 + 5x + 2$.

a Find $\frac{dy}{dx}$ in terms of x. **(2 marks)**

b The points P and Q lie on C. The gradient of C at both P and Q is 2. The x-coordinate of P is 3.

i Find the x-coordinate of Q. **(3 marks)**

ii Find an equation for the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants. **(3 marks)**

iii If this tangent intersects the coordinate axes at the points R and S, find the length of RS, giving your answer as a surd. **(3 marks)**

13 A curve has equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. Find the equations of the tangent and the normal to the curve at the point where $x = 2$.

E/P **14** The normals to the curve $2y = 3x^3 - 7x^2 + 4x$, at the points $O(0, 0)$ and $A(1, 0)$, meet at the point N.

a Find the coordinates of N. **(7 marks)**

b Calculate the area of triangle OAN. **(3 marks)**

E/P **15** A curve C has equation $y = x^3 - 2x^2 - 4x - 1$ and cuts the y-axis at a point P. The line L is a tangent to the curve at P, and cuts the curve at the point Q. Show that the distance PQ is $2\sqrt{17}$. **(7 marks)**

E **16** A curve has equation $y = x^3 - 6x^2 + 9x$. Find the coordinates of its **local** maximum. **(4 marks)**

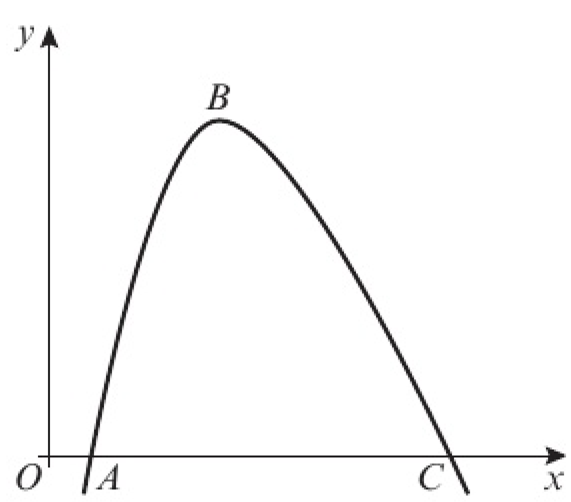
E **17** The diagram shows part of the curve with equation $y = f(x)$, where:

$$f(x) = 200 - \frac{250}{x} - x^2, x > 0$$

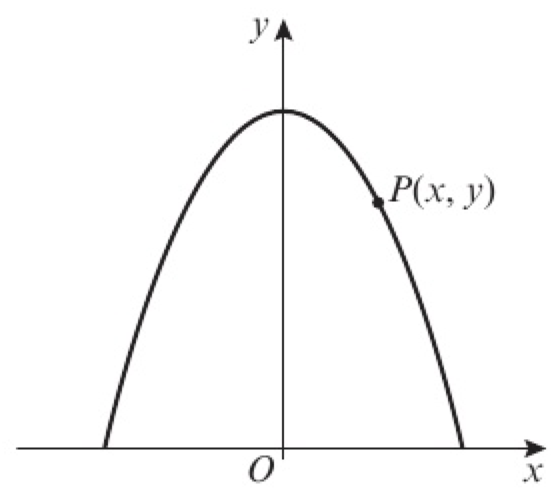
The curve cuts the x-axis at the points A and C. The point B is the maximum point of the curve.

a Find $f'(x)$. **(3 marks)**

b Use your answer to part **a** to calculate the coordinates of B. **(4 marks)**



E/P **18** The diagram shows part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y > 0$.
The point $P(x, y)$ lies on the curve and O is the origin.



a Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$. **(3 marks)**

Taking $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$:

b Find the values of x for which $f'(x) = 0$. **(4 marks)**

c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. **(4 marks)**

Summary of key points

1 The gradient of a curve at a given point is defined as the gradient of the tangent to the curve at that point.

2 The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\frac{dy}{dx}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

3 For all real values of n, and for a constant a:

- If $f(x) = x^n$ then $f'(x) = nx^{n-1}$
- If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$
- If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
- If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

4 For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by:

$$\frac{dy}{dx} = 2ax + b$$

5 If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$.

6 The tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

$$y - f(a) = f'(a)(x - a)$$

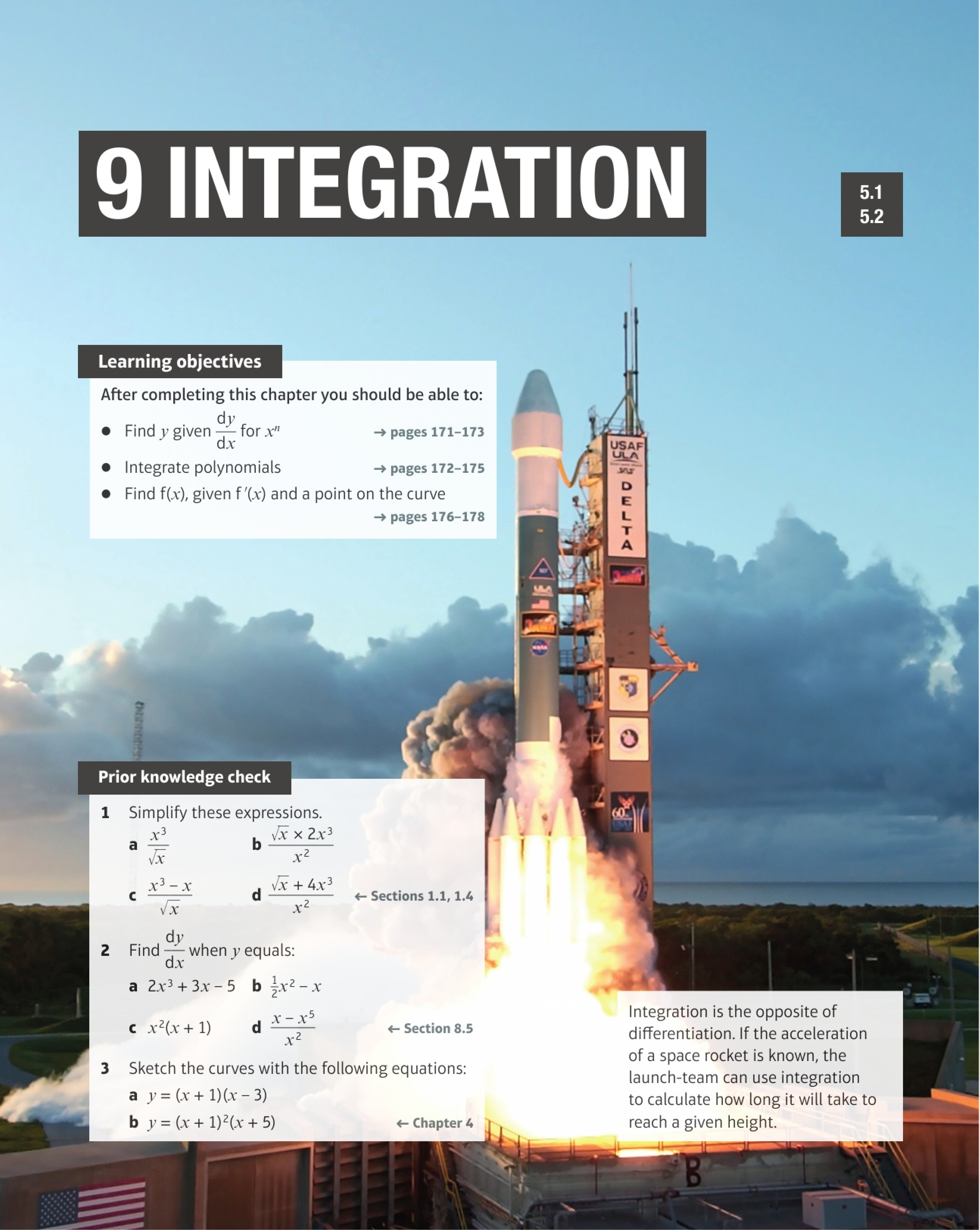
7 The normal to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

8 Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$.

9 INTEGRATION

5.1
5.2



Learning objectives

After completing this chapter you should be able to:

- Find y given $\frac{dy}{dx}$ for x^n → pages 171–173
- Integrate polynomials → pages 172–175
- Find $f(x)$, given $f'(x)$ and a point on the curve → pages 176–178

Prior knowledge check

1 Simplify these expressions.

a $\frac{x^3}{\sqrt{x}}$ **b** $\frac{\sqrt{x} \times 2x^3}{x^2}$

c $\frac{x^3 - x}{\sqrt{x}}$ **d** $\frac{\sqrt{x} + 4x^3}{x^2}$ ← Sections 1.1, 1.4

2 Find $\frac{dy}{dx}$ when y equals:

a $2x^3 + 3x - 5$ **b** $\frac{1}{2}x^2 - x$

c $x^2(x + 1)$ **d** $\frac{x - x^5}{x^2}$ ← Section 8.5

3 Sketch the curves with the following equations:

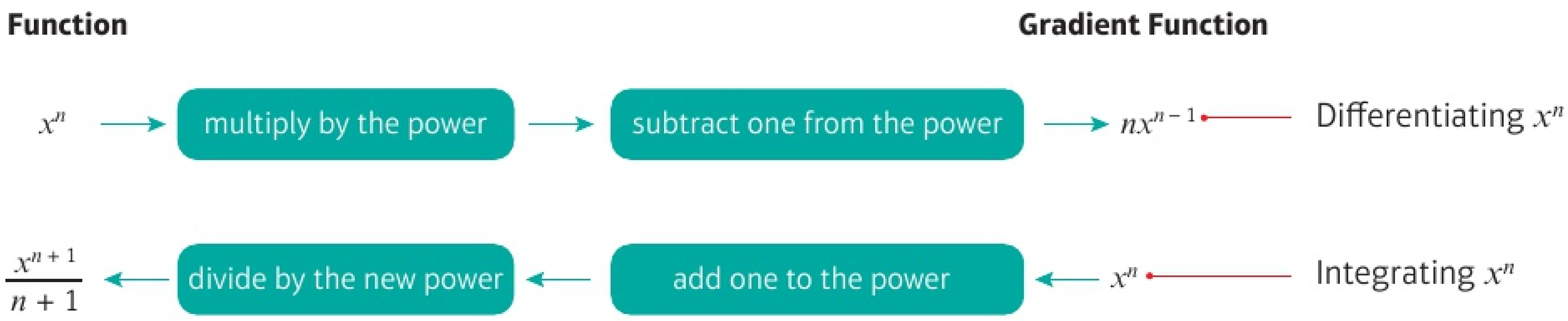
a $y = (x + 1)(x - 3)$

b $y = (x + 1)^2(x + 5)$ ← Chapter 4

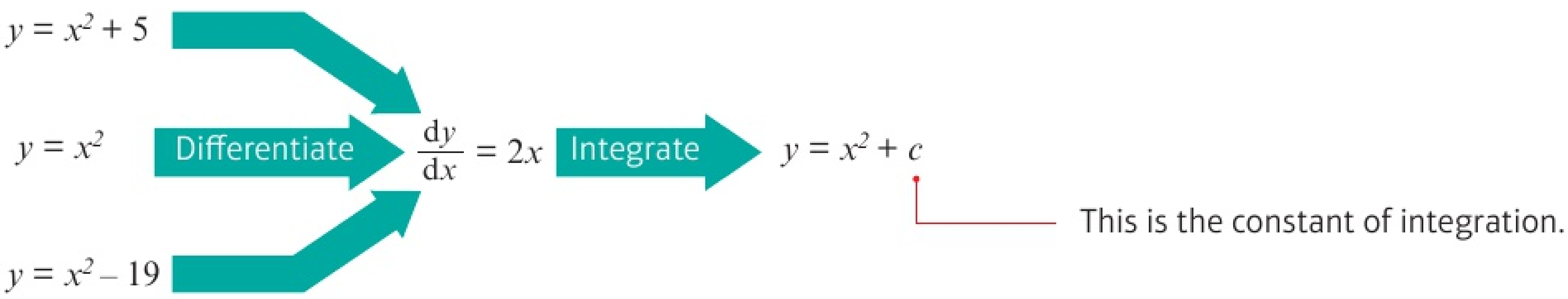
Integration is the opposite of differentiation. If the acceleration of a space rocket is known, the launch-team can use integration to calculate how long it will take to reach a given height.

9.1 Integrating x^n

Integration is the reverse process of differentiation:



Constant terms disappear when you differentiate. This means that when you differentiate functions that only **differ** in the constant term, they will all differentiate to give the same function. To allow for this, you need to add a **constant of integration** at the end of a function when you integrate.



- If $\frac{dy}{dx} = xn$, then $y = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$
- If $f'(x) = xn$, then $f(x) = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$

Example 1

Find y for the following:

a $\frac{dy}{dx} = x^4$ **b** $\frac{dy}{dx} = x^{-5}$

a $y = \frac{x^5}{5} + c$

b $y = \frac{x^{-4}}{-4} + c = -\frac{1}{4}x^{-4} + c$

Use $y = \frac{1}{n+1}x^{n+1} + c$ with $n = 4$.

Don't forget to add c.

Remember, adding 1 to the power gives $-5 + 1 = -4$.

Divide by the new power (-4) and add c.

Example 2

Find $f(x)$ for the following:

a $f'(x) = 3x^{\frac{1}{2}}$ **b** $f'(x) = 3$

a $f(x) = 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c$

Remember $3 \div \frac{3}{2} = 3 \times \frac{2}{3} = 2$
Simplify your answer.

b $f'(x) = 3 = 3x^0$

So $f(x) = 3 \times \frac{x^1}{1} + c = 3x + c$

$x^0 = 1$, so 3 can be written as $3x^0$.

You can integrate a function in the form kx^n by integrating x^n and multiplying the integral by k.

- **If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.**
- **Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.**
- **When integrating polynomials, apply the rule of integration separately to each term.**

Watch out You don't need to multiply the constant term (c) by k.

Example 3 SKILLS INTERPRETATION

Given $\frac{dy}{dx} = 6x + 2x^{-3} - 3x^{\frac{1}{2}}$, find y.

$y = \frac{6x^2}{2} + \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$

$= 3x^2 - x^{-2} - 2x^{\frac{3}{2}} + c$

Apply the rule of integration to each term of the expression and add c.
Now simplify each term and remember to add c.

Exercise 9A SKILLS INTERPRETATION

1 Find an expression for y when $\frac{dy}{dx}$ is the following:

a x^5 b $10x^4$ c $-x^{-2}$ d $-4x^{-3}$ e $x^{\frac{2}{3}}$ f $4x^{\frac{1}{2}}$
g $-2x^6$ h $x^{-\frac{1}{2}}$ i $5x^{-\frac{3}{2}}$ j $6x^{\frac{1}{3}}$ k $36x^{11}$ l $-14x^{-8}$
m $-3x^{-\frac{2}{3}}$ n -5 o $6x$ p $2x^{-0.4}$

2 Find y when $\frac{dy}{dx}$ is given by the following expressions. In each case simplify your answer.

a $x^3 - \frac{3}{2}x^{-\frac{1}{2}} - 6x^{-2}$ b $4x^3 + x^{-\frac{2}{3}} - x^{-2}$ c $4 - 12x^{-4} + 2x^{-\frac{1}{2}}$
d $5x^{\frac{2}{3}} - 10x^4 + x^{-3}$ e $-\frac{4}{3}x^{-\frac{4}{3}} - 3 + 8x$ f $5x^4 - x^{-\frac{3}{2}} - 12x^{-5}$

3 Find $f(x)$ when $f'(x)$ is given by the following expressions. In each case simplify your answer.

a $12x + \frac{3}{2}x^{-\frac{3}{2}} + 5$ b $6x^5 + 6x^{-7} - \frac{1}{6}x^{-\frac{7}{6}}$ c $\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$
d $10x^4 + 8x^{-3}$ e $2x^{-\frac{1}{3}} + 4x^{-\frac{5}{3}}$ f $9x^2 + 4x^{-3} + \frac{1}{4}x^{-\frac{1}{2}}$

E/P 4 Find y given that $\frac{dy}{dx} = (2x + 3)^2$. **(4 marks)**

Problem-solving Start by expanding the brackets.

(E) **5** Find $f(x)$ given that $f'(x) = 3x^{-2} + 6x^{\frac{1}{2}} + x - 4$. **(4 marks)**

Challenge

Find y when $\frac{dy}{dx} = (2\sqrt{x} - x^2)\left(\frac{3+x}{x^5}\right)$.

9.2 Indefinite integrals

You can use the symbol $\int$ to represent the process of **integration**.

- $\int f'(x)dx = f(x) + c$

You can write the process of integrating x^n as follows:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

The elongated S means integrate.

The expression to be integrated.

The dx tells you to integrate with respect to x.

When you are integrating a polynomial function, you can integrate the terms one at a time.

- $\int (f'(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

Example 4 SKILLS PROBLEM SOLVING

Find:

a $\int (x^{\frac{1}{2}} + 2x^3)\,dx$ **b** $\int (x^{-\frac{3}{2}} + 2)\,dx$ **c** $\int (p^2x^{-2} + q)\,dx$ **d** $\int (4t^2 + 6)\,dt$

a $\int (x^{\frac{1}{2}} + 2x^3)dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{2x^4}{4} + c$

First apply the rule term by term.

$= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^4 + c$

Simplify each term.

b $\int (x^{-\frac{3}{2}} + 2)dx = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 2x + c$

$= -2x^{-\frac{1}{2}} + 2x + c$

Remember $-\frac{3}{2} + 1 = -\frac{1}{2}$ and the integral of the constant 2 is $2x$.

c $\int (p^2x^{-2} + q)dx = \frac{p^2}{-1}x^{-1} + qx + c$

$= -p^2x^{-1} + qx + c$

The dx tells you to integrate with respect to the variable x, so any other letters must be treated as constants.

d $\int (4t^2 + 6)dt = \frac{4t^3}{3} + 6t + c$

The dt tells you that this time you must integrate with respect to t.

Use the rule for integrating x^n but replace x with t:

If $\frac{dy}{dt} = kt^n$, then $y = \frac{k}{n+1}t^{n+1} + c, n \neq -1$.

Before you integrate, you need to ensure that each term of the expression is in the form kx^n, where k and n are real numbers.

Example 5

Find:

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)dx$ **b** $\int x\left(x^2 + \frac{2}{x}\right)dx$ **c** $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)dx$

a $\int\left(\frac{2}{x^3} - 3\sqrt{x}\right)dx$

$= \int(2x^{-3} - 3x^{\frac{1}{2}})dx$ — First write each term in the form x^n.

$= \frac{2}{-2}x^{-2} - \frac{3}{\frac{3}{2}}x^{\frac{3}{2}} + c$ — Apply the rule term by term.

$= -x^{-2} - 2x^{\frac{3}{2}} + c$ — Simplify each term.

$= -\frac{1}{x^2} - 2\sqrt{x^3} + c$ — Sometimes it is helpful to write the answer in the same form as the question.

b $\int x\left(x^2 + \frac{2}{x}\right)dx$

$= \int(x^3 + 2)dx$ — First multiply out the bracket.

$= \frac{x^4}{4} + 2x + c$ — Then apply the rule to each term.

c $\int\left((2x)^2 + \frac{\sqrt{x}+5}{x^2}\right)dx$

$= \int\left(4x^2 + \frac{x^{\frac{1}{2}}}{x^2} + \frac{5}{x^2}\right)dx$ — Simplify $(2x)^2$ and write $\sqrt{x}$ as $x^{\frac{1}{2}}$.

$= \int(4x^2 + x^{-\frac{3}{2}} + 5x^{-2})dx$ — Write each term in the form x^n.

$= \frac{4}{3}x^3 + \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{5x^{-1}}{-1} + c$ — Apply the rule term by term.

$= \frac{4}{3}x^3 - 2x^{-\frac{1}{2}} - 5x^{-1} + c$

$= \frac{4}{3}x^3 - \frac{2}{\sqrt{x}} - \frac{5}{x} + c$ — Finally simplify the answer.

Exercise 9B SKILLS PROBLEM SOLVING

1 Find the following integrals:

a $\int x^3\,dx$ **b** $\int x^7\,dx$ **c** $\int 3x^{-4}\,dx$ **d** $\int 5x^2\,dx$

2 Find the following integrals:

a $\int(x^4 + 2x^3)dx$ **b** $\int(2x^3 - x^2 + 5x)dx$ **c** $\int(5x^{\frac{3}{2}} - 3x^2)dx$

3 Find the following integrals:

a $\int(4x^{-2} + 3x^{-\frac{1}{2}})dx$ **b** $\int(6x^{-2} - x^{\frac{1}{2}})dx$ **c** $\int(2x^{-\frac{3}{2}} + x^2 - x^{-\frac{1}{2}})dx$

4 Find the following integrals:

a $\int(4x^3 - 3x^{-4} + 2)\,dx$ **b** $\int(x + x^{-\frac{1}{2}} + x^{-\frac{3}{2}})\,dx$

c $\int(px^4 + 2q + 3x^{-2})\,dx$

Hint In **Q4** part **c** you are integrating with respect to x, so treat p and q as constants.

5 Find the following integrals:

a $\int(3t^2 - t^{-2})\,dt$ **b** $\int(2t^2 - 3t^{-\frac{3}{2}} + 1)\,dt$ **c** $\int(pt^3 + q^2 + pr^3)\,dt$

6 Find the following integrals:

a $\int\frac{(2x^3 + 3)}{x^2}\,dx$ **b** $\int(2x + 3)^2\,dx$ **c** $\int(2x + 3)\sqrt{x}\,dx$

7 Find $\int f(x)\,dx$ when $f(x)$ is given by the following:

a $\left(x + \frac{1}{x}\right)^2$ **b** $(\sqrt{x} + 2)^2$ **c** $\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right)$

8 Find the following integrals:

a $\int\left(x^{\frac{2}{3}} + \frac{4}{x^3}\right)dx$ **b** $\int\left(\frac{2 + x}{x^3} + 3\right)dx$ **c** $\int(x^2 + 3)(x - 1)\,dx$

d $\int\frac{(2x + 1)^2}{\sqrt{x}}\,dx$ **e** $\int\left(3 + \frac{\sqrt{x} + 6x^3}{x}\right)dx$ **f** $\int\sqrt{x}(\sqrt{x} + 3)^2\,dx$

9 Find the following integrals:

a $\int\left(\frac{A}{x^2} - 3\right)dx$ **b** $\int\left(\sqrt{Px} + \frac{2}{x^3}\right)dx$ **c** $\int\left(\frac{p}{x^2} + q\sqrt{x} + r\right)dx$

(E) **10** Given that $f(x) = \frac{6}{x^2} + 4\sqrt{x} - 3x + 2$, $x > 0$, find $\int f(x)\,dx$. **(5 marks)**

(E) **11** Find $\int\left(8x^3 + 6x - \frac{3}{\sqrt{x}}\right)dx$, giving each term in its simplest form. **(4 marks)**

(E/P) **12** **a** Show that $(2 + 5\sqrt{x})^2$ can be written as $4 + k\sqrt{x} + 25x$, where k is a constant to be found. **(2 marks)**

b Hence find $\int(2 + 5\sqrt{x})^2\,dx$. **(3 marks)**

(E) **13** Given that $y = 3x^5 - \frac{4}{\sqrt{x}}$, $x > 0$, find $\int y\,dx$ in its simplest form. **(3 marks)**

(E/P) **14** $\int\left(\frac{p}{2x^2} + pq\right)dx = \frac{2}{x} + 10x + c$ **(5 marks)**

Find the value of p and the value of q.

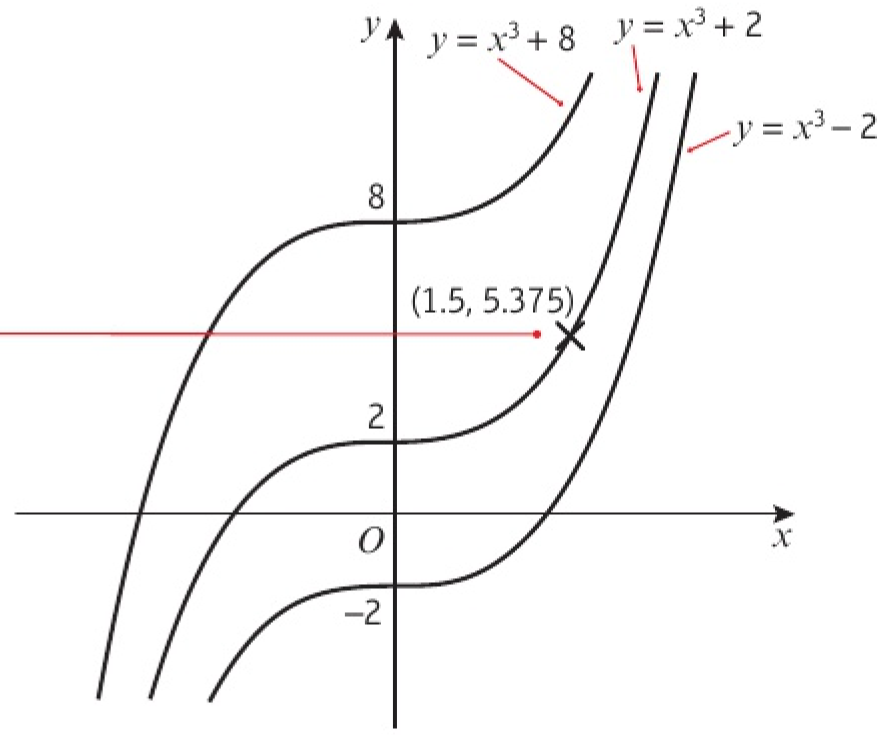
Problem-solving

Integrate the expression on the left-hand side, treating p and q as constants, then compare the result with the right-hand side.

9.3 Finding functions

You can find the constant of integration, c, when you are given (i) any point (x, y) that the curve of the function passes through or (ii) any value that the function takes. For example, if $\frac{dy}{dx} = 3x^2$ then $y = x^3 + c$. There are infinitely many curves with this equation, depending on the value of c.

Only one of these curves passes through this point. Choosing a point on the curve determines the value of c.



- **To find the constant of integration, c**
  - Integrate the function
  - Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$, into the integrated function
  - Solve the equation to find c

Example 6 SKILLS ANALYSIS

The curve C with equation $y = f(x)$ passes through the point (4, 5). Given that $f'(x) = \frac{x^2 - 2}{\sqrt{x}}$, find the equation of C.

$f'(x) = \frac{x^2 - 2}{\sqrt{x}} = x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}$ — First write $f'(x)$ in a form suitable for integration.

So $f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$ — Integrate as normal and don't forget the $+ c$.

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$

But $f(4) = 5$ — Use the fact that the curve passes through (4, 5).

So $5 = \frac{2}{5} \times 2^5 - 4 \times 2 + c$ — Remember $4^{\frac{5}{2}} = 2^5$.

$5 = \frac{64}{5} - 8 + c$

$5 = \frac{24}{5} + c$

So $c = \frac{1}{5}$ — Solve for c.

So $y = \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + \frac{1}{5}$ — Finally write down the equation of the curve.

Online Explore the solution using technology.

Exercise 9C SKILLS ANALYSIS

1 Find the equation of the curve with the given derivative of y with respect to x that passes through the given point:

a $\frac{dy}{dx} = 3x^2 + 2x$ point (2, 10)

b $\frac{dy}{dx} = 4x^3 + \frac{2}{x^3} + 3$ point (1, 4)

c $\frac{dy}{dx} = \sqrt{x} + \frac{1}{4}x^2$ point (4, 11)

d $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$ point (4, 0)

e $\frac{dy}{dx} = (x + 2)^2$ point (1, 7)

f $\frac{dy}{dx} = \frac{x^2 + 3}{\sqrt{x}}$ point (0, 1)

2 The curve C, with equation $y = f(x)$, passes through the point (1, 2), and $f'(x) = 2x^3 - \frac{1}{x^2}$. Find the equation of C in the form $y = f(x)$.

3 The gradient of a particular curve is given by $\frac{dy}{dx} = \frac{\sqrt{x} + 3}{x^2}$.
Given that the curve passes through the point (9, 0), find an equation of the curve.

E 4 The curve with equation $y = f(x)$ passes through the point (−1, 0).
Given that $f'(x) = 9x^2 + 4x - 3$, find $f(x)$. **(5 marks)**

E/P 5 $\frac{dy}{dx} = 3x^{-\frac{1}{2}} - 2x\sqrt{x}$, $x > 0$.
Given that $y = 10$ at $x = 4$, find y in terms of x, giving each term in its simplest form. **(7 marks)**

E/P 6 **a** Given that $\frac{6x + 5x^{\frac{3}{2}}}{\sqrt{x}}$ can be written in the form $6x^p + 5x^q$,
write down the value of p and the value of q. **(2 marks)**

b Given that $\frac{dy}{dx} = \frac{6x + 5x^{\frac{3}{2}}}{\sqrt{x}}$, and that $y = 100$ when $x = 9$,
find y in terms of x, simplifying the coefficient of each term. **(5 marks)**

P 7 The displacement of a particle at time t is given by the function $f(t)$, where $f(0) = 0$.
Given that the velocity of the particle is given by $f'(t) = 10 - 5t$,

a find $f(t)$

b determine the displacement of the particle when $t = 3$.

Problem-solving

You don't need any specific knowledge of mechanics to answer this question. You are told that the displacement of the particle at time t is given by $f(t)$.

P **8** The height, in metres, of an arrow fired horizontally from the top of a castle is modelled by the function f(t), where f(0) = 35. Given that f′(t) = −9.8t,

a find f(t)

b determine the height of the arrow when $t = 1.5$

c write down the height of the castle according to this model

d estimate the time it will take the arrow to hit the ground

e state one assumption used in your calculation.

Challenge

1 A set of curves, where each curve passes through the origin, has equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$ … where $f'_n(x) = f_{n-1}(x)$ and $f_1(x) = x^2$.

a Find $f_2(x)$ and $f_3(x)$.

b Suggest an expression for $f_n(x)$.

2 A set of curves, with equations $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$, … all pass through the point (0, 1) and they are related by the property $f'_n(x) = f_{n-1}(x)$ and $f_1(x) = 1$. Find $f_2(x)$, $f_3(x)$ and $f_4(x)$.

Chapter review 9

SKILLS EXECUTIVE FUNCTION

1 Find:

a $\int(x + 1)(2x - 5)\,dx$ **b** $\int(x^{\frac{1}{3}} + x^{-\frac{1}{3}})\,dx$

2 The gradient of a curve is given by $f'(x) = x^2 - 3x - \frac{2}{x^2}$. Given that the curve passes through the point (1, 1), find the equation of the curve in the form $y = f(x)$.

3 Find:

a $\int(8x^3 - 6x^2 + 5)\,dx$ **b** $\int(5x + 2)x^{\frac{1}{2}}\,dx$

P **4** Given $y = \frac{(x + 1)(2x - 3)}{\sqrt{x}}$, find $\int y\,dx$.

P **5** Given that $\frac{dx}{dt} = (t + 1)^2$ and that $x = 0$ when $t = 2$, find the value of x when $t = 3$.

E/P **6** Given that $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$,

a show that $y = x^{\frac{2}{3}} + Ax^{\frac{1}{3}} + B$, where A and B are constants to be found **(2 marks)**

b hence find $\int y\,dx$. **(3 marks)**

E/P **7** Given that $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$ $(x > 0)$,

a find $\frac{dy}{dx}$ **(2 marks)**

b find $\int y\,dx$. **(3 marks)**

P **8** $\int\left(\frac{a}{3x^3} - ab\right)dx = -\frac{2}{3x^2} + 14x + c$

Find the value of a and the value of b.

P **9** A rock is dropped off a cliff. The height in metres of the rock above the ground after t seconds is given by the function $f(t)$. Given that $f(0) = 70$ and $f'(t) = -9.8t$, find the height of the rock above the ground after 3 seconds.

P **10** A cyclist is travelling along a straight road. The distance in metres of the cyclist from a fixed point after t seconds is modelled by the function $f(t)$, where $f'(t) = 5 + 2t$ and $f(0) = 0$.

a Find an expression for $f(t)$.

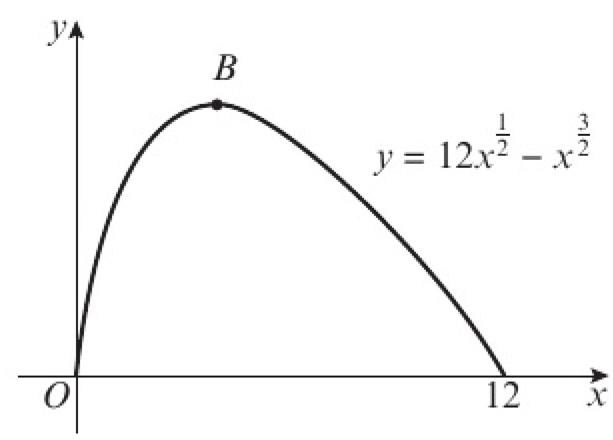
b Calculate the time taken for the cyclist to travel 100 m.

E **11** Consider the function $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$, $x > 0$.

a Find $\frac{dy}{dx}$. **(2 marks)**

b Find $\int y\,dx$. **(3 marks)**

E/P **12** The diagram shows a sketch of the curve with equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$, for $0 \leqslant x \leqslant 12$.



a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b At the point B on the curve the tangent to the curve is parallel to the x-axis. Find the coordinates of the point B. **(2 marks)**

E **13** Given that $f(x) = \frac{9}{x^2} - 8\sqrt{x} + 4x - 5$, $x > 0$, find $\int f(x)dx$. **(5 marks)**

E/P **14** $f'(x) = \frac{(2 - x^2)^3}{x^2}$, $x \neq 0$

a Show that $f'(x) = 8x^{-2} - 12 + Ax^2 + Bx^4$, where A and B are constants to be found. **(3 marks)**

b Given that the point $(-2, 9)$ lies on the curve with equation $y = f(x)$, find $f(x)$. **(5 marks)**

Challenge

Points (1, 4) and (2, 12) lie on a curve whose gradient is given by $\frac{dy}{dx} = 6x^2 - 6x + k$, where k is a constant.

a Find the value of k.

b Hence, or otherwise, write down the equation of the curve.

Summary of key points

1 If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$.

Using function notation, if $f'(x) = x^n$, then $f(x) = \frac{1}{n+1}x^{n+1} + c$, $n \neq -1$.

2 If $\frac{dy}{dx} = kx^n$, then $y = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.

Using function notation, if $f'(x) = kx^n$, then $f(x) = \frac{k}{n+1}x^{n+1} + c$, $n \neq -1$.

When integrating polynomials, apply the rule of integration separately to each term.

3 $\int f'(x)dx = f(x) + c$

4 $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$

5 To find the constant of integration, c:

- Integrate the function
- Substitute the values (x, y) of a point on the curve, or the value of the function at a given point $f(x) = k$ into the integrated function
- Solve the equation to find c

Review exercise 2

(E) **1** Find the equation of the line which passes through the points $A(-2, 8)$ and $B(4, 6)$, in the form $ax + by + c = 0$. **(3)**

← Section 5.2

(E) **2** The line l passes through the point $(9, -4)$ and has gradient $\frac{1}{3}$. Find an equation for l, in the form $ax + by + c = 0$, where a, b and c are integers. **(3)**

← Section 5.2

(E/P) **3** The points $A(0, 3)$, $B(k, 5)$ and $C(10, 2k)$, where k is a constant, lie on the same straight line. Find the two possible values of k. **(5)**

← Section 5.1

(E) **4** The line l_1 has equation $y = 3x - 6$.
The line l_2 is perpendicular to l_1 and passes through the point $(6, 2)$.

a Find an equation for l_2 in the form $y = mx + c$, where m and c are constants. **(3)**

The lines l_1 and l_2 intersect at the point C.

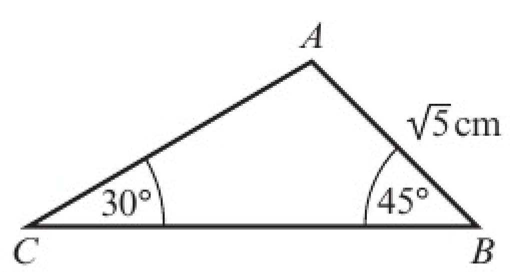
b Use algebra to find the coordinates of C. **(2)**

The lines l_1 and l_2 cross the x-axis at the points A and B respectively.

c Calculate the exact area of triangle ABC. **(4)**

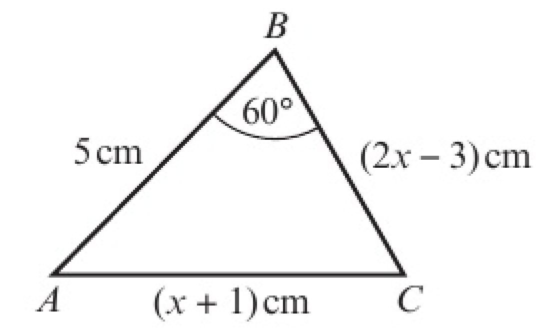
← Sections 5.3, 5.4

(E) **5** The diagram shows triangle ABC, with $AB = \sqrt{5}$ cm, $\angle ABC = 45°$ and $\angle BCA = 30°$. Find the exact length of AC. **(3)**



← Section 6.2

(E/P) **6** The diagram shows triangle ABC, with $AB = 5$ cm, $BC = (2x - 3)$ cm, $CA = (x + 1)$ cm and $\angle ABC = 60°$.



a Show that x satisfies the equation $x^2 - 8x + 16 = 0$. **(3)**

b Find the value of x. **(1)**

c Calculate the area of the triangle, giving your answer to 3 significant figures. **(2)**

← Section 6.4

(E/P) **7** Ship B is 8 km, on a bearing of 030°, from ship A.
Ship C is 12 km, on a bearing of 140°, from ship B.

a Calculate the distance of ship C from ship A. **(4)**

b Calculate the bearing of ship C from ship A. **(3)**

← Section 6.4

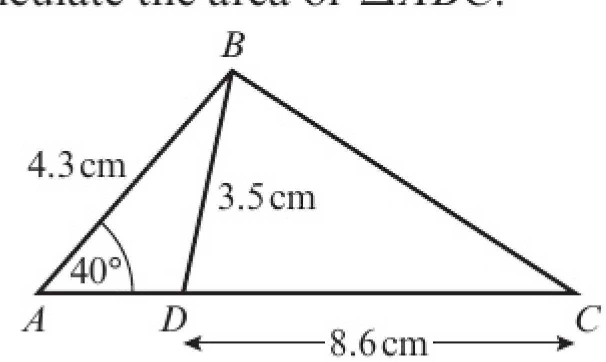
(E/P) **8** The triangle ABC has vertices $A(-2, 4)$, $B(6, 10)$ and $C(16, 10)$.

a Prove that ABC is an isosceles triangle. **(2)**

b Calculate the size of $\angle ABC$. **(3)**

← Sections 5.4, 6.4

(E/P) **9** The diagram shows ΔABC.
Calculate the area of ΔABC. **(6)**



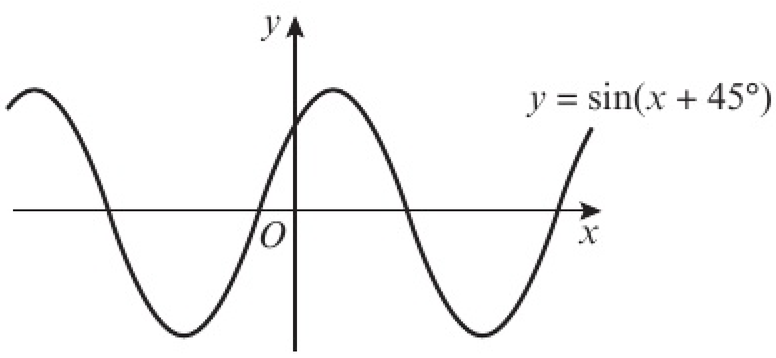
← Section 6.4

E **10** **a** On the same set of axes, in the interval $0 \leqslant x \leqslant 360°$, sketch the graphs of $y = \tan(x - 90°)$ and $y = \sin x$. Label clearly any points at which the graphs cross the coordinate axes. **(5)**

b Hence write down the number of solutions of the equation $\tan(x - 90°) = \sin x$ in the interval $0 \leqslant x \leqslant 360°$. **(1)**

← Section 6.6

E **11** The graph shows the curve $y = \sin(x + 45°)$, $-360° \leqslant x \leqslant 360°$.

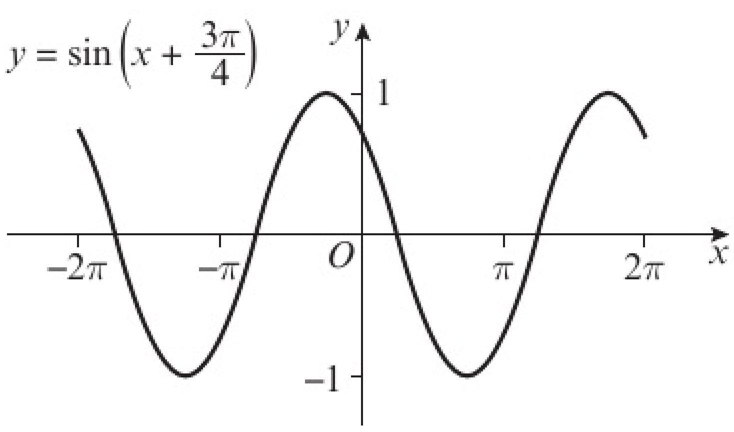


a Write down the coordinates of each point where the curve crosses the x-axis. **(2)**

b Write down the coordinates of the point where the curve crosses the y-axis. **(1)**

← Section 6.6

E **12** The diagram shows the curve with equation $y = \sin\left(x + \frac{3\pi}{4}\right)$, $-2\pi \leqslant x \leqslant 2\pi$.



Calculate the coordinates of the points at which the curve meets the coordinate axes. **(3)**

← Section 6.6

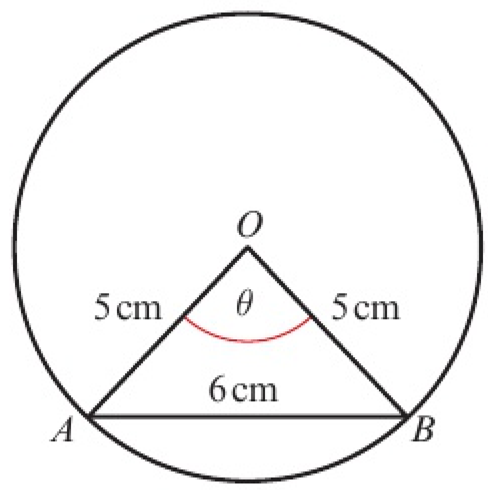
E **13** **a** Sketch, for $0 \leqslant x \leqslant 2\pi$, the graph of $y = \cos\left(x - \frac{\pi}{3}\right)$. **(2)**

b Write down the exact coordinates of the points where the graph meets the coordinate axes. **(3)**

c Solve, for $0 \leqslant x \leqslant 2\pi$, the equation $\cos\left(x - \frac{\pi}{3}\right) = -0.27$, giving your answers in radians to 2 decimal places. **(5)**

← Section 6.6

E **14** In the diagram, A and B are points on the circumference of a circle centre O and radius 5 cm.
$\angle AOB = \theta$ radians
$AB = 6$ cm

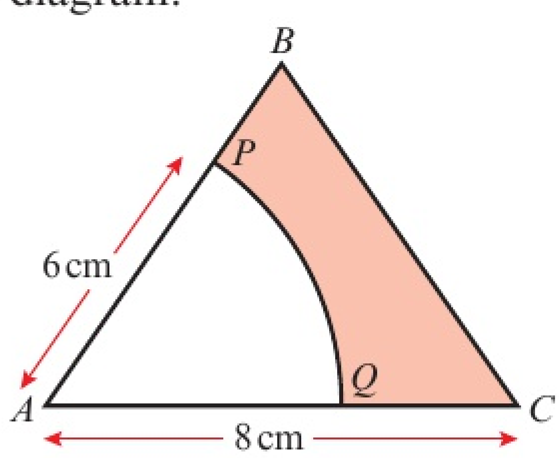


a Find the value of θ. **(2)**

b Calculate the length of the minor arc AB to 3 s.f. **(2)**

← Section 7.2

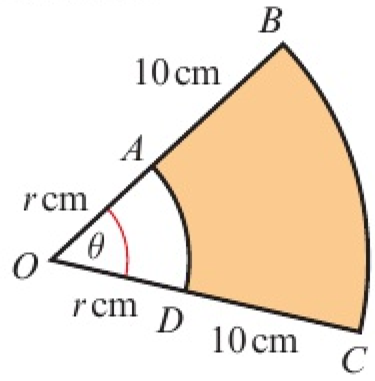
E/P **15** In the diagram, ABC is an equilateral triangle with side 8 cm. PQ is an arc of a circle with centre A and radius 6 cm. Find the perimeter of the shaded region in the diagram.



(5)

← Section 7.2

E/P **16** In the diagram, AD and BC are arcs of circles with centre O, such that $OA = OD = r$ cm, $AB = DC = 10$ cm and $\angle BOC = \theta$ radians.

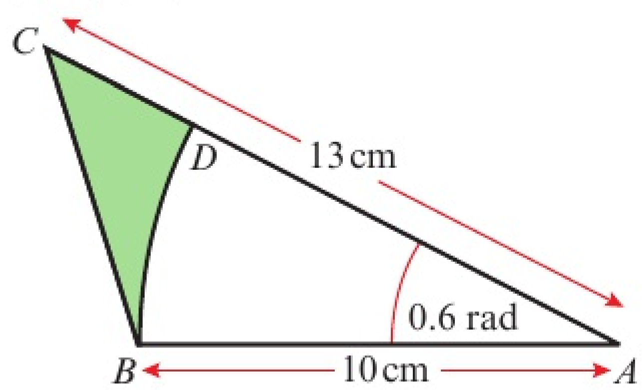


a Given that the area of the shaded region is $40\,\text{cm}^2$, show that $r = \frac{4}{\theta} - 5$. **(4)**

b Given also that $r = 6\theta$, calculate the perimeter of the shaded region. **(6)**

← Sections 7.2, 7.3

E/P **17** In the diagram, $AB = 10$ cm, $AC = 13$ cm and $\angle CAB = 0.6$ radians.
BD is an arc of a circle centre A and radius 10 cm.

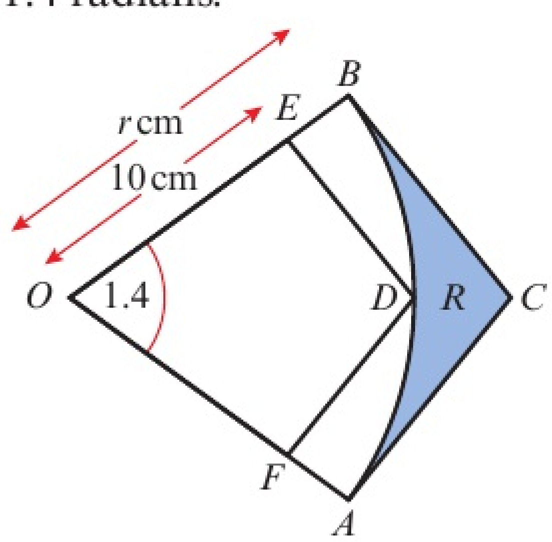


a Calculate the length of the arc BD. **(2)**

b Calculate the shaded area in the diagram to 1 d.p. **(3)**

← Sections 7.2, 7.3

E/P **18** The diagram shows the sector OAB of a circle with centre O, radius r cm and angle 1.4 radians.



The lines AC and BC are tangent to the circle with centre O. OEB and OFA are straight lines. The line ED is parallel to BC and the line FD is parallel to AC.

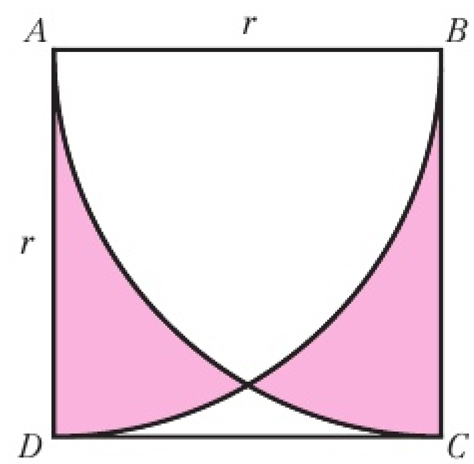
a Find the area of sector OAB, giving your answer to 1 decimal place. **(4)**

The region R is bounded by the arc AB and the lines AC and CB.

b Find the perimeter of R, giving your answer to 1 decimal place. **(6)**

← Sections 7.2, 7.3

E/P **19** The diagram shows a square, $ABCD$, with side length r, and two arcs of circles with centres A and B.



Show that the area of the shaded region is $\frac{r^2}{12}(3\sqrt{3} - \pi)$. **(5)**

← Sections 7.2, 7.3

E/P **20** Prove, from first principles, that the derivative of $5x^2$ is $10x$. **(4)**

← Section 8.2

E **21** Given that $y = 4x^3 - 1 + 2x^{\frac{1}{2}}$, $x > 0$, find $\frac{dy}{dx}$. **(2)**

← Section 8.5

E/P **22** The curve C has equation $y = 4x + 3x^{\frac{3}{2}} - 2x^2$, $x > 0$.

a Find an expression for $\frac{dy}{dx}$. **(2)**

b Show that the point $P(4, 8)$ lies on C. **(1)**

c Show that an equation of the normal to C at point P is $3y = x + 20$. **(2)**

The normal to C at P cuts the x-axis at point Q.

d Find the length PQ, giving your answer in simplified surd form. **(2)**

← Sections 8.5, 8.6

E/P **23** The curve C has equation

$y = 4x^2 + \frac{5-x}{x}$, $x \neq 0$.

The point P on C has x-coordinate 1.

a Show that the value of $\frac{dy}{dx}$ at P is 3. **(3)**

b Find an equation of the tangent to C at P. **(3)**

This tangent meets the x-axis at the point $(k, 0)$.

c Find the value of k. **(1)**

← Section 8.6

E/P **24** $f(x) = \frac{(2x+1)(x+4)}{\sqrt{x}}$, $x > 0$

a Show that $f(x)$ can be written in the form $Px^{\frac{3}{2}} + Qx^{\frac{1}{2}} + Rx^{-\frac{1}{2}}$, stating the values of the constants P, Q and R. **(2)**

b Find $f'(x)$. **(3)**

c A curve has equation $y = f(x)$. Show that the tangent to the curve at the point where $x = 1$ is parallel to the line with equation $2y = 11x + 3$. **(3)**

← Section 8.6

E **25** Given that $y = 3x^2 + 4\sqrt{x}$, $x > 0$, find

a $\frac{dy}{dx}$ **(2)**

b $\frac{d^2y}{dx^2}$ **(2)**

c $\int y\,dx$ **(3)**

← Sections 8.7, 9.2

E **26** The curve C with equation $y = f(x)$ passes through the point $(5, 65)$.

Given that $f'(x) = 6x^2 - 10x - 12$,

a use integration to find $f(x)$ **(3)**

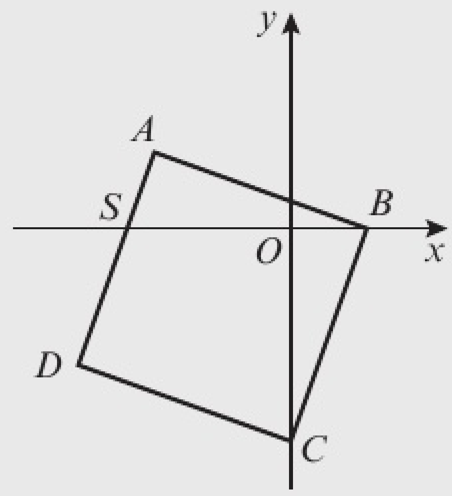
b hence show that $f(x) = x(2x+3)(x-4)$ **(2)**

c sketch C, showing the coordinates of the points where C crosses the x-axis. **(3)**

← Sections 4.1, 9.3

Challenge SKILLS CREATIVITY

1 The diagram shows a square $ABCD$ on a set of coordinate axes. The square intersects the x-axis at the points B and S, and the equation of the line which passes through B and C is $y = 3x - 12$.

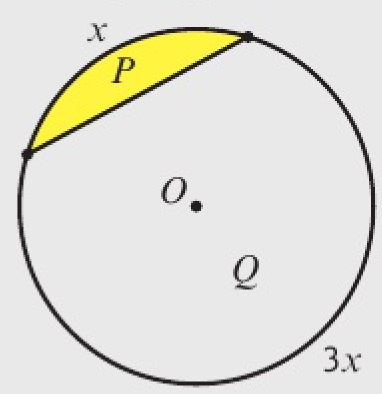


a Calculate the area of the square.

b Find the coordinates of S.

← Sections 5.2, 5.4

2 A chord of a circle, centre O and radius r, divides the circumference in the ratio 1 : 3, as shown in the diagram. Find the ratio of the area of region P to the area of region Q.



← Section 7.3

3 The graph of the cubic function $y = f(x)$ has turning points at $(-3, 76)$ and $(2, -49)$.

a Show that $f'(x) = k(x^2 + x - 6)$, where k is a constant.

b Express $f(x)$ in the form $ax^3 + bx^2 + cx + d$, where a, b, c and d are real constants to be found.

← Section 9.3

Exam practice

Mathematics
International Advanced Subsidiary/ Advanced Level Pure Mathematics 1

Time: 1 hour 30 minutes
You must have: Mathematical Formulae and Statistical Tables, Calculator
Answer ALL questions

1 Simplify

a $(2a\sqrt{7b})^2$ **(2)**

b $(2a^2\sqrt[3]{6b})^3$ **(2)**

c Simplify $\frac{1-\sqrt{7}}{3-\sqrt{7}}$, giving your answer in the form $p + q\sqrt{7}$, where p and q are rational numbers. **(3)**

2 Solve the simultaneous equations

$y + 3x + 1 = 0$

$y^2 + 11x^2 + 3x = 0$ **(6)**

3 The point (9, 5) lies on a curve with equation $y = \text{f}(x)$.

Given that $\text{f}'(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$, $x > 0$, find $\text{f}(x)$.

Give each term in your answer in its simplest form. **(6)**

4 $\frac{x^2 - 10}{5\sqrt{x}}$ can be written in the form $Ax^p + Bx^q$, where A, B, p and q are constants to be determined.

a Find A, B, p and q. **(5)**

Hence or otherwise, given that $y = \frac{x^2 - 10}{5\sqrt{x}}$ and $x > 0$,

b find $\frac{\text{d}y}{\text{d}x}$, giving your answer in its simplest form **(4)**

c find $\int y\,\text{d}x$, giving your answer in its simplest form. **(5)**

5 Figure 1 shows a sketch graph with equation $y = f(x)$.
The curve passes through the origin O and has a maximum point A at $(-4, 9)$ and a minimum point B at $(2, -3)$.

On separate diagrams, sketch the curves with equations

a $y = 2f(x)$ **(3)**

b $y = f(x) - 9$ **(3)**

c $y = f(x + 2)$ **(3)**

d $y = f(2x)$ **(4)**

On each diagram, show clearly the coordinates of the maximum and the minimum points.

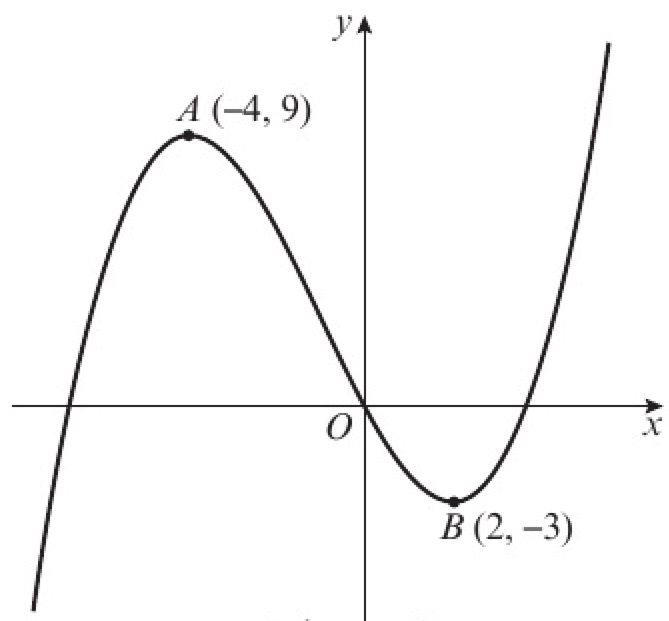


Figure 1

6 Given that $y = 2x^2 + 9\sqrt[3]{x} + \dfrac{x^3 - 6}{4\sqrt{x}}$, $x > 0$, find $\dfrac{dy}{dx}$. Give each term in your answer in simplified form. **(6)**

7 The equation $x^2 + kx + (k + 4) = 0$, where k is a constant, has distinct real roots.

a Show that $k^2 - 4k - 16 > 0$. **(3)**

b Find the set of possible values of k, giving your answer in surd form. **(5)**

8 $x - 2py = -3$

$3x - 2y + q = -4$

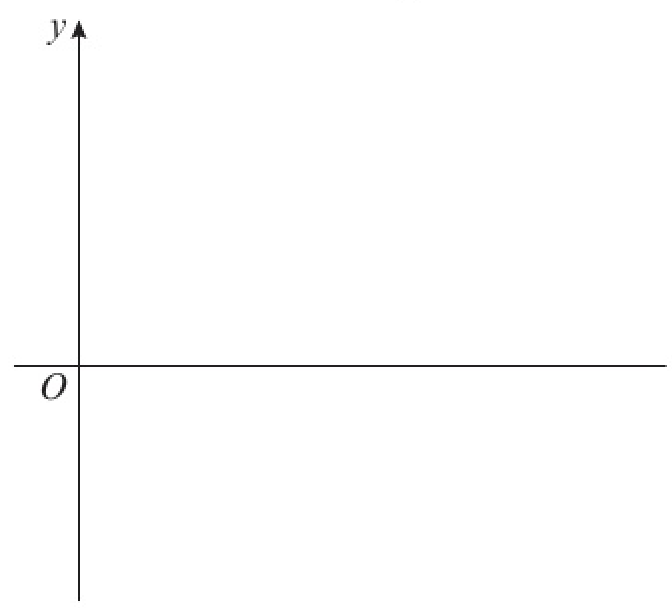
are simultaneous equations where p and q are constants.

The solution to these simultaneous equations is $x = 1$, $y = q$. Find the values of p and q. **(6)**

9 A curve C has equation $y = \cos\left(x - \frac{\pi}{4}\right)$, $0 \leqslant x \leqslant 2\pi$.

a On the axes below, sketch the curve C. **(2)**

b Write down the exact coordinates of all the points at which the curve C meets or intersects the x-axis and the y-axis. **(3)**



10 A straight line has gradient 3 and passes through the point (2, 1).
At point P it intersects a line with equation $2x - 3y + 6 = 0$.
Find the coordinates of point P. **(4)**

TOTAL FOR PAPER: 75 MARKS

GLOSSARY

algebraic (as opposed to **graphical**) – representing mathematical information symbolically (using numbers, letters and other signs) rather than visually (using graphs)

appropriate suitable, sensible. It is *appropriate* to give an answer to 1 decimal place if it is not an integer

arc part of the circumference of a circle

axis (plural **axes**) either of the two lines by which the positions of points are measured on a graph

base in the expression 4^3, 4 is the *base*

base (of a triangle) any side, but usually the side parallel to the bottom of the drawing

bounded enclosed

cancel (out) for example: $\frac{ab}{ac} = \frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$

chord a line segment joining two points on the circumference

circumference the perimeter of a circle

collect to bring together terms of the same type

common factor – a quantity that will divide without remainder into two or more other quantities

congruent (triangles) having sides of equal lengths (as well as equal angles)

consider think about

constant a term that does not include a variable. In the expression $3x^2 + 4x + 2$, the term 2 is a *constant*

coordinate grid the set of points defined by the axes on a graph

cubic expressions in which the highest power is 3

cuboid a six-sided solid whose sides are all rectangles, placed at right angles. Can be, but not necessarily, a cube

deduce to reach a logical conclusion. If $x + 2 = 3$, we can *deduce* that $x = 1$

denominator the lower part of a fraction. B in $\frac{A}{B}$

determine to find out. By analysing the equation of a curve, we can *determine* whether or not it is a circle. If you are asked to determine a quantity, that quantity is to be determined

difference the result of subtraction

dimensions the size of something, usually given as its length, width and height

discard to cancel or remove (for example, to ignore a solution that doesn't make sense)

displacement change of position

distinct different

elimination the removal (*elimination*) of a variable in two simultaneous equations

evaluate calculate the value of. If $x = 2$, we can *evaluate* $4x$ to 8

expand to multiply out terms in brackets

explore to make discoveries about a region by visiting various parts of it

express rearrange or write in a different form. We can *express* 3×3 as 3^2

expression any group of terms that represents something mathematical. $x + 1$, $4x^2$ and $y + 3x^3 + 2x^2$ are *expressions*

factor a number that divides into another number exactly

factorise to rewrite an expression using brackets. We *factorise* $x^2 + 3x + 2$ to get $(x + 1)(x + 2)$

formula a mathematical expression or equation with a particular meaning

formalise to clearly describe the details of a process

fractional less than 1

given that (statement)… if we know that (statement) is true…

gradient slope

graph (noun or verb) a mathematical function represented visually

hence means the same as 'therefore' or 'so': $2x = 6$, *hence* $x = 3$

illustrate to help to show the meaning of something, usually by visual means

indefinite (integrals) not having a fixed value

index (plural **indices**) the power to which a quantity is raised, shown as a small number (superscript). For example, the number 3 in x^3

inequality the opposite of an equation. The equals sign (=) is replaced by an *in*equality sign: $>$, $\geqslant$, $<$, $\leqslant$, <> or $\neq$. For example '$x \geqslant 1$' means 'x is greater than or equal to 1'

integer a whole number

intersection the point at which two or more curves cross (*intersect*)

isosceles (triangle) having two equal sides, hence two equal angles

label (noun or verb) text helping to explain a picture

lie on if a point *lies on* a curve, the curve runs through that point

like terms have the same variable raised to the same power. In the expression $4x^3 + 5x^2 + 3x^2 + 5x - 3$, $5x^2$ and $3x^2$ are *like terms*

line segment a line between two points

linear where the variables have the power 1. Hence $y = 2x + 3$ is *linear* but $y = x^2$ and $y = \frac{1}{x}$ (i.e. $y = x^{-1}$) are not. A linear function can be represented by a straight line

local (as opposed to **global**) in the context of *nearby* values, but not necessarily of values *further away*

manipulate (an expression) to change it into a more convenient form

member (of, for example, a set) belonging to

minor, **major arc** the *minor* arc is measured by the shorter route around the circumference

numerator the upper part of a fraction. A in $\frac{A}{B}$

overlap if two or more things *overlap*, part of one thing covers part of another

perimeter the whole length of the border around an area or shape

period how often a curve repeats itself. *sin* θ has a period of 360° (2π)

polynomial usually has three or more terms, e.g. $x^2 + 5x + 2$

power if a number is increased to the *power* of three, four, five etc., it is multiplied by itself three, four, five etc. times. A to the power of 2 is $A \times A$, written as A^2

prime an integer that has no integer factors. For example, 21 = 7 × 3, so 21 is *not* prime

product 2 × 3 = 6, so 6 is the *product* of 2 and 3

projectile motion the movement of an object after it has been thrown or fired

quadrant a quarter of a circle

quadratic an expression such as $x^2 + 2x + 3$, containing a variable to the power of 2 (but no higher power)

quadrilateral has four sides

rational any real number, including integers, that can be written as a fraction, even if it produces a repeating decimal. Many functions are **irrational**: most roots, logarithms and trigonometric functions, and the constants π and e

radius (plural **radii**) the line segment joining the centre to the circumference of a circle

real any number representing a quantity. 3, –3, $\sqrt{3}$, $\frac{1}{3}$, log 3, sin 3, π and e are all *real*. $\sqrt{-3}$ and ln(–3) are both *un*real. Not all real numbers are *rational*

rearrange to put terms in a different order

reciprocal the *reciprocal* of $\frac{31}{4}$ is $\frac{4}{31}$, and vice versa

region an area of a graph enclosed by curves or lines

roots (of an equation) the set of all possible solutions. A quadratic equation usually has two *roots*

scale extent to which a shape has been stretched

scalene (triangle) having no equal sides or equal angles

sector is bounded by an arc and two radii

segment is bounded by an arc and a chord

set a group of similar items

set notation using symbols such as { } ∩ ∪ ℝ

shade to make part of a picture or drawing darker or otherwise different from its surroundings

significant figures digits counted from the first (non-zero) digit on the left

simplify to replace an expression with a simpler, usually shorter, one

sketch (noun or verb) a drawing that explains something without necessarily being accurate

square number the number formed when an integer is multiplied by itself. 9 is a *square number* (the original integer being 3)

square root (of x) the number which, when multiplied by itself, equals x

steep at a large angle

stretch make larger or smaller. The latter meaning is exclusive to maths

substitute to replace something (e.g. a *variable*) with something else (e.g. a *value*). Let $y = x + 1$. If we *substitute* $x = 2$, we find that $y = 2 + 1 = 3$

subtend the angle *subtended* by an arc is the angle at the centre formed by the two radii

surd $\sqrt{4} = 2$, but $\sqrt{2}$ can't be resolved into a rational number. We call $\sqrt{2}$ a surd

symmetrical, **symmetry** two shapes are *symmetrical* if one can be transformed into the other by reflecting, rotating or stretching

term a separate part of a mathematical expression. The function $ax^2 + bx + c$ has three *terms:* (i) ax^2, (ii) bx and (iii) c.

transform translate, rotate, reflect or stretch

translate move

trigonometric concerned with the relationship between the angles and the sides of triangles

undefined not having a meaning or a value, for example the result of division by zero

vertex (plural **vertices**) where two lines meet at an angle, especially in a shape such as a triangle

ANSWERS

CHAPTER 1

Prior knowledge check

1 a $2m^2n + 3mn^2$ b $6x^2 - 12x - 10$
2 a 2^8 b 2^4 c 2^6
3 a $3x + 12$ b $10 - 15x$ c $12x - 30y$
4 a 8 b $2x$ c xy
5 a $2x$ b $10x$ c $\frac{5x}{3}$

Exercise 1A

1 a x^7 b $6x^5$ c k d $2p^2$
e x f y^{10} g $5x^2$ h p^2
i $2a^3$ j $2p$ k $6a^9$ l $3a^2b^3$
m $27x^8$ n $24x^{11}$ o $63a^{12}$ p $32y^6$
q $4a^6$ r $6a^{12}$

2 a $9x - 18$ b $x^2 + 9x$
c $-12y + 9y^2$ d $xy + 5x$
e $-3x^2 - 5x$ f $-20x^2 - 5x$
g $4x^2 + 5x$ h $-15y + 6y^3$
i $-10x^2 + 8x$ j $3x^3 - 5x^2$
k $4x - 1$ l $2x - 4$
m $9d^2 - 2c$ n $13 - r^2$
o $3x^3 - 2x^2 + 5x$ p $14y^2 - 35y^3 + 21y^4$
q $-10y^2 + 14y^3 - 6y^4$ r $4x + 10$
s $11x - 6$ t $7x^2 - 3x + 7$
u $-2x^2 + 26x$ v $-9x^3 + 23x^2$

3 a $3x^3 + 5x^5$ b $3x^4 - x^6$ c $\frac{x^3}{2} - x$
d $4x^2 + \frac{5}{2}$ e $\frac{7x^6}{5} + x$ f $3x^4 - \frac{5x^2}{3}$

Exercise 1B

1 a $x^2 + 11x + 28$
b $x^2 - x - 6$
c $x^2 - 4x + 4$
d $2x^2 + 3x - 2xy - 3y$
e $4x^2 + 11xy - 3y^2$
f $6x^2 - 10xy - 4y^2$
g $2x^2 - 11x + 12$
h $9x^2 + 12xy + 4y^2$
i $4x^2 + 6x + 16xy + 24y$
j $2x^2 + 3xy + 5x + 15y - 25$
k $3x^2 - 4xy - 8x + 4y + 5$
l $2x^2 + 5x - 7xy - 4y^2 - 20y$
m $x^2 + 2x + 2xy + 6y - 3$
n $2x^2 + 15x + 2xy + 12y + 18$
o $13y - 4x + 12 - 4y^2 + xy$
p $12xy - 4y^2 + 3y + 15x + 10$
q $5xy - 20y - 2x^2 + 11x - 12$
r $22y - 4y^2 - 5x + xy - 10$

2 a $5x^2 - 15x - 20$
b $14x^2 + 7x - 70$
c $3x^2 - 18x + 27$
d $x^3 - xy^2$
e $6x^3 + 8x^2 + 3x^2y + 4xy$
f $x^2y - 4xy - 5y$
g $12x^2y + 6xy - 8xy^2 - 4y^2$
h $19xy - 35y - 2x^2y$
i $10x^3 - 4x^2 + 5x^2y - 2xy$
j $x^3 + 3x^2y - 2x^2 + 6xy - 8x$
k $2x^2y + 9xy + xy^2 + 5y^2 - 5y$
l $6x^2y + 4xy^2 + 2y^2 - 3xy - 3y$
m $2x^3 + 2x^2y - 7x^2 + 3xy - 15x$
n $24x^3 - 6x^2y - 26x^2 + 2xy + 6x$
o $6x^3 + 15x^2 - 3x^2y - 18xy^2 - 30xy$
p $x^3 + 6x^2 + 11x + 6$
q $x^3 + x^2 - 14x - 24$
r $x^3 - 3x^2 - 13x + 15$
s $x^3 - 12x^2 + 47x - 60$
t $2x^3 - x^2 - 5x - 2$
u $6x^3 + 19x^2 + 11x - 6$
v $18x^3 - 15x^2 - 4x + 4$
w $x^3 - xy^2 - x^2 + y^2$
x $8x^3 - 36x^2y + 54xy^2 - 27y^3$

3 $2x^2 - xy + 29x - 7y + 24$
4 $(4x^3 + 12x^2 + 5x - 6)\,\text{cm}^3$
5 $a = 12, b = 32, c = 3, d = -5$

Challenge
$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Exercise 1C

1 a $4(x + 2)$ b $6(x - 4)$
c $5(4x + 3)$ d $2(x^2 + 2)$
e $4(x^2 + 5)$ f $6x(x - 3)$
g $x(x - 7)$ h $2x(x + 2)$
i $x(3x - 1)$ j $2x(3x - 1)$
k $5y(2y - 1)$ l $7x(5x - 4)$
m $x(x + 2)$ n $y(3y + 2)$
o $4x(x + 3)$ p $5y(y - 4)$
q $3xy(3y + 4x)$ r $2ab(3 - b)$
s $5x(x - 5y)$ t $4xy(3x + 2y)$
u $5y(3 - 4z^2)$ v $6(2x^2 - 5)$
w $xy(y - x)$ x $4y(3y - x)$

2 a $x(x + 4)$ b $2x(x + 3)$
c $(x + 8)(x + 3)$ d $(x + 6)(x + 2)$
e $(x + 8)(x - 5)$ f $(x - 6)(x - 2)$
g $(x + 2)(x + 3)$ h $(x - 6)(x + 4)$
i $(x - 5)(x + 2)$ j $(x + 5)(x - 4)$
k $(2x + 1)(x + 2)$ l $(3x - 2)(x + 4)$
m $(5x - 1)(x - 3)$ n $2(3x + 2)(x - 2)$
o $(2x - 3)(x + 5)$ p $2(x^2 + 3)(x^2 + 4)$
q $(x + 2)(x - 2)$ r $(x + 7)(x - 7)$
s $(2x + 5)(2x - 5)$ t $(3x + 5y)(3x - 5y)$
u $4(3x + 1)(3x - 1)$ v $2(x + 5)(x - 5)$
w $2(3x - 2)(x - 1)$ x $3(5x - 1)(x + 3)$

3 a $x(x^2 + 2)$ b $x(x^2 - x + 1)$
c $x(x^2 - 5)$ d $x(x + 3)(x - 3)$
e $x(x - 4)(x + 3)$ f $x(x + 5)(x + 6)$
g $x(x - 1)(x - 6)$ h $x(x + 8)(x - 8)$
i $x(2x + 1)(x - 3)$ j $x(2x + 3)(x + 5)$
k $x(x + 2)(x - 2)$ l $3x(x + 4)(x + 5)$

4 $(x^2 + y^2)(x + y)(x - y)$
5 $x(3x + 5)(2x - 1)$

Challenge
$(x - 1)(x + 1)(2x + 3)(2x - 3)$

Exercise 1D

1 a x^5 b x^{-2} c x^4 d x^3
e x^5 f $12x^0 = 12$ g $3x^{\frac{1}{2}}$ h $5x$
i $6x^{-1}$ j $x^{\frac{5}{6}}$ k $x^{\frac{17}{6}}$ l $x^{\frac{1}{6}}$

Online Worked solutions are available in SolutionBank.

2 **a** 5 **b** 729 **c** 3 **d** $\frac{1}{16}$
e $\frac{1}{3}$ **f** $\frac{-1}{125}$ **g** 1 **h** 216
i $\frac{125}{64}$ **j** $\frac{9}{4}$ **k** $\frac{5}{6}$ **l** $\frac{64}{49}$

3 **a** $8x^5$ **b** $\frac{5}{x^2} - \frac{2}{x^3}$ **c** $5x^4$
d $\frac{1}{x^2} + 4$ **e** $\frac{2}{x^3} + \frac{1}{x^2}$ **f** $\frac{8}{27}x^6$
g $\frac{3}{x} - 5x^2$ **h** $\frac{1}{3x^2} + \frac{1}{5x}$

4 **a** 3 **b** $\frac{16}{\sqrt[3]{x}}$

5 **a** $\frac{x}{2}$ **b** $\frac{32}{x^6}$

Exercise 1E

1 **a** $2\sqrt{7}$ **b** $6\sqrt{2}$ **c** $5\sqrt{2}$ **d** $4\sqrt{2}$
e $3\sqrt{10}$ **f** $\sqrt{3}$ **g** $\sqrt{3}$ **h** $6\sqrt{5}$
i $7\sqrt{2}$ **j** $12\sqrt{7}$ **k** $-3\sqrt{7}$ **l** $9\sqrt{5}$
m $23\sqrt{5}$ **n** 2 **o** $19\sqrt{3}$

2 **a** $2\sqrt{3} + 3$ **b** $3\sqrt{5} - \sqrt{15}$
c $4\sqrt{2} - \sqrt{10}$ **d** $6 + 2\sqrt{5} - 3\sqrt{2} - \sqrt{10}$
e $6 - 2\sqrt{7} - 3\sqrt{3} + \sqrt{21}$ **f** $13 + 6\sqrt{5}$
g $8 - 6\sqrt{3}$ **h** $5 - 2\sqrt{3}$
i $3 + 5\sqrt{11}$

3 $3\sqrt{3}$

Exercise 1F

1 **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{11}}{11}$ **c** $\frac{\sqrt{2}}{2}$
d $\frac{\sqrt{5}}{5}$ **e** $\frac{1}{2}$ **f** $\frac{1}{4}$
g $\frac{\sqrt{13}}{13}$ **h** $\frac{1}{3}$

2 **a** $\frac{1 - \sqrt{3}}{-2}$ **b** $\sqrt{5} - 2$ **c** $\frac{3 + \sqrt{7}}{2}$
d $3 + \sqrt{5}$ **e** $\frac{\sqrt{5} + \sqrt{3}}{2}$ **f** $\frac{(3 - \sqrt{2})(4 + \sqrt{5})}{11}$
g $5(\sqrt{5} - 2)$ **h** $5(4 + \sqrt{14})$ **i** $\frac{11(3 - \sqrt{11})}{-2}$
j $\frac{5 - \sqrt{21}}{-2}$ **k** $\frac{14 - \sqrt{187}}{3}$ **l** $\frac{35 + \sqrt{1189}}{6}$
m -1

3 **a** $\frac{11 + 6\sqrt{2}}{49}$ **b** $9 - 4\sqrt{5}$ **c** $\frac{44 + 24\sqrt{2}}{49}$
d $\frac{81 - 30\sqrt{2}}{529}$ **e** $\frac{13 + 2\sqrt{2}}{161}$ **f** $\frac{7 - 3\sqrt{3}}{11}$

4 $-\frac{7}{4} + \frac{\sqrt{5}}{4}$

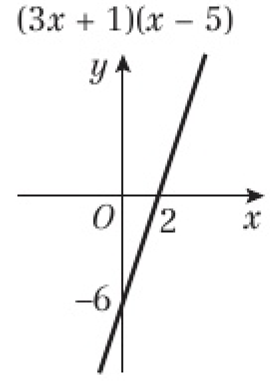
Chapter review 1

1 **a** y^8 **b** $6x^7$ **c** $32x$ **d** $12b^9$
2 **a** $x^2 - 2x - 15$ **b** $6x^2 - 19x - 7$
c $6x^2 - 2xy + 19x - 5y + 10$
3 **a** $x^3 + 3x^2 - 4x$ **b** $x^3 + 6x^2 - 13x - 42$
c $6x^3 - 5x^2 - 17x + 6$
4 **a** $15y + 12$ **b** $15x^2 - 25x^3 + 10x^4$
c $16x^2 + 13x$ **d** $9x^3 - 3x^2 + 4x$
5 **a** $x(3x + 4)$ **b** $2y(2y + 5)$
c $x(x + y + y^2)$ **d** $2xy(4y + 5x)$
6 **a** $(x + 1)(x + 2)$ **b** $3x(x + 2)$
c $(x - 7)(x + 5)$ **d** $(2x - 3)(x + 1)$
e $(5x + 2)(x - 3)$ **f** $(1 - x)(6 + x)$
7 **a** $2x(x^2 + 3)$ **b** $x(x + 6)(x - 6)$
c $x(2x - 3)(x + 5)$
8 **a** $3x^6$ **b** 2 **c** $6x^2$ **d** $\frac{1}{2}x^{-\frac{1}{3}}$
9 **a** $\frac{4}{9}$ **b** $\frac{3375}{4913}$
10 **a** $\frac{\sqrt{7}}{7}$ **b** $4\sqrt{5}$
11 **a** $21\,877$
b $(5x + 6)(7x - 8)$
When $x = 25$, $5x + 6 = 131$ and $7x - 8 = 167$; both 131 and 167 are prime numbers.
12 **a** $3\sqrt{2} + \sqrt{10}$ **b** $10 + 2\sqrt{3} - 5\sqrt{5} - \sqrt{15}$
c $24 - 6\sqrt{7} - 4\sqrt{2} + \sqrt{14}$
13 **a** $\frac{\sqrt{3}}{3}$ **b** $\sqrt{2} + 1$ **c** $-3\sqrt{3} - 6$
d $\frac{30 - \sqrt{851}}{-7}$ **e** $7 - 4\sqrt{3}$ **f** $\frac{23 + 8\sqrt{7}}{81}$
14 **a** $b = -4$ and $c = -5$ **b** $(x + 3)(x - 5)(x + 1)$
15 **a** $\frac{1}{4}x$ **b** $256x^{-3}$
16 $\frac{5}{\sqrt{75} - \sqrt{50}} = \frac{1}{\sqrt{3} - \sqrt{2}} = \sqrt{3} + \sqrt{2}$
17 $-36 + 10\sqrt{11}$
18 $x(1 + 8x)(1 - 8x)$
19 $y = 6x + 3$
20 $4\sqrt{3}$
21 $(3 - \sqrt{3})$ cm
22 $\frac{4 - 4x^{\frac{1}{2}} + x^1}{x^{\frac{1}{2}}} = 4x^{-\frac{1}{2}} - 4 + x^{\frac{1}{2}}$
23 $\frac{11}{2}$
24 $4x^{\frac{5}{2}} + x^2$, $a = \frac{5}{2}$ $b = 2$

Challenge
a $a - b$
b $\frac{(\sqrt{1} - \sqrt{2}) + (\sqrt{2} - \sqrt{3}) + \ldots + (\sqrt{24} - \sqrt{25})}{-1} = \sqrt{25} - \sqrt{1} = 4$

CHAPTER 2

Prior knowledge check

1 **a** $x = -5$ **b** $x = 3$
c $x = 5$ or $x = -5$ **d** 16 or 0
2 **a** $(x + 3)(x + 5)$ **b** $(x + 5)(x - 2)$
c $(3x + 1)(x - 5)$ **d** $(x - 20)(x + 20)$
3 **a**


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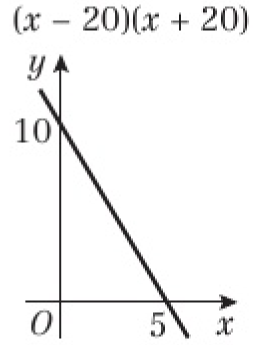
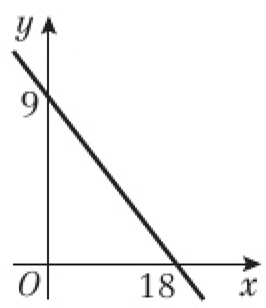
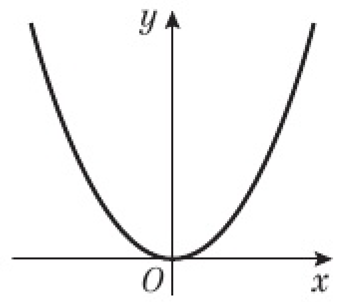


3 c 

d 

4 a $x < 3$ b $x \geqslant 9$ c $x \leqslant 2.5$ d $x > -7$

Exercise 2A

1 a $x = -1$ or $x = -2$ b $x = -1$ or $x = -4$
c $x = -5$ or $x = -2$ d $x = 3$ or $x = -2$
e $x = 3$ or $x = 5$ f $x = 4$ or $x = 5$
g $x = 6$ or $x = -1$ h $x = 6$ or $x = -2$

2 a $x = 0$ or $x = 4$ b $x = 0$ or $x = 25$
c $x = 0$ or $x = 2$ d $x = 0$ or $x = 6$
e $x = -\frac{1}{2}$ or $x = -3$ f $x = -\frac{1}{3}$ or $x = \frac{3}{2}$
g $x = -\frac{2}{3}$ or $x = \frac{3}{2}$ h $x = \frac{3}{2}$ or $x = \frac{5}{2}$

3 a $x = \frac{1}{3}$ or $x = -2$ b $x = 3$ or $x = 0$
c $x = 13$ or $x = 1$ d $x = 2$ or $x = -2$
e $x = \pm\sqrt{\frac{5}{3}}$ f $x = 3 \pm \sqrt{13}$
g $x = \dfrac{1 \pm \sqrt{11}}{3}$ h $x = 1$ or $x = -\frac{7}{6}$
i $x = -\frac{1}{2}$ or $x = \frac{7}{3}$ j $x = 0$ or $x = -\frac{11}{6}$

4 $x = 4$

5 $x = -1$ or $x = -\frac{2}{25}$

Exercise 2B

1 a $x = \frac{1}{2}(-3 \pm \sqrt{5})$ b $x = \frac{1}{2}(3 \pm \sqrt{17})$
c $x = -3 \pm \sqrt{3}$ d $x = \frac{1}{2}(5 \pm \sqrt{33})$
e $x = \frac{1}{3}(-5 \pm \sqrt{31})$ f $x = \frac{1}{2}(1 \pm \sqrt{2})$
g $x = 2$ or $x = -\frac{1}{4}$ h $x = \frac{1}{11}(-1 \pm \sqrt{78})$

2 a $x = -0.586$ or $x = -3.41$ b $x = 7.87$ or $x = 0.127$
c $x = 0.765$ or $x = -11.8$ d $x = 8.91$ or $x = -1.91$
e $x = 0.105$ or $x = -1.90$ f $x = 3.84$ or $x = -2.34$
g $x = 4.77$ or $x = 0.558$ h $x = 4.89$ or $x = -1.23$

3 a $x = -6$ or $x = -2$ b $x = 1.09$ or $x = -10.1$
c $x = 9.11$ or $x = -0.110$ d $x = -\frac{1}{2}$ or $x = -2$
e $x = 1$ or $x = -9$ f $x = 1$
g $x = 4.68$ or $x = -1.18$ h $x = 3$ or $x = 5$

4 Area $= \frac{1}{2}(2x)(x + (x + 10)) = 50\,\text{m}^2$
So $x^2 + 5x - 25 = 0$
Using the quadratic formula:
$x = \frac{1}{2}(-5 \pm 5\sqrt{5})$
Height $= 2x = 5(\sqrt{5} - 1)\,\text{m}$

Challenge
$x = 13$

Exercise 2C

1 a $(x + 2)^2 - 4$ b $(x - 3)^2 - 9$
c $(x - 8)^2 - 64$ d $(x + \frac{1}{2})^2 - \frac{1}{4}$
e $(x - 7)^2 - 49$

2 a $2(x + 4)^2 - 32$ b $3(x - 4)^2 - 48$
c $5(x + 2)^2 - 20$ d $2(x - \frac{5}{4})^2 - \frac{25}{8}$
e $-2(x - 2)^2 + 8$

3 a $2(x + 2)^2 - 7$ b $5(x - \frac{3}{2})^2 - \frac{33}{4}$
c $3(x + \frac{1}{3})^2 - \frac{4}{3}$ d $-4(x + 2)^2 + 26$
e $-8(x - \frac{1}{8})^2 + \frac{81}{8}$

4 $a = \frac{3}{2}$, $b = \frac{15}{4}$

5 $A = 6$, $B = 0.04$, $C = -10$

Exercise 2D

1 a $x = -3 \pm 2\sqrt{2}$ b $x = -6 \pm \sqrt{33}$
c $x = -2 \pm \sqrt{6}$ d $x = 5 \pm \sqrt{30}$

2 a $x = \frac{1}{2}(-3 \pm \sqrt{15})$ b $x = \frac{1}{5}(-4 \pm \sqrt{26})$
c $x = \frac{1}{8}(1 \pm \sqrt{129})$ d $x = \frac{1}{2}(-3 \pm \sqrt{39})$

3 a $p = -7$, $q = -48$
b $(x - 7)^2 = 48$
$x = 7 \pm \sqrt{48} = 7 \pm 4\sqrt{3}$
$r = 7$, $s = 4$

4 $x^2 + 2bx + c = (x + b)^2 - b^2 + c$
$(x + b)^2 = b^2 - c$
$x = -b \pm \sqrt{b^2 - c}$

Challenge

a $ax^2 + 2bx + c = 0$
$$x^2 + \frac{2b}{a}x + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{a}\right)^2 - \frac{b^2}{a^2} + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{a}\right)^2 = \frac{b^2 - ac}{a^2}$$
$$x = -\frac{b}{a} \pm \sqrt{\frac{b^2 - ac}{a^2}}$$

b $ax^2 + bx + c = 0$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 2E

1 a 8 b 7 c 3 d 10.5 e 0
f 0 g 25 h 2 i 7

2 $a = 4$ or $a = -2$

3 a $\frac{2}{3}$ b 2 and -9 c -10 and 4
d 12 and -12 e 0, -5 and -7 f 0, 3 and -8

4 $x = 3$ and $x = 2$

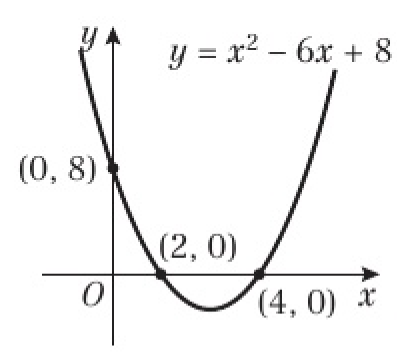
5 $x = 0$, 2.5 and 6

6 a $(x - 1)^2 + 1$
$p = -1$, $q = 1$
b Squared terms are always $\geqslant 0$, so the minimum value is $0 + 1 = 1$

7 a -2 and -1 b 2, -2, $2\sqrt{2}$ and $-2\sqrt{2}$
c -1 and $\frac{1}{3}$ d $\frac{1}{2}$ and 1
e 4 and 25 f 8 and -27

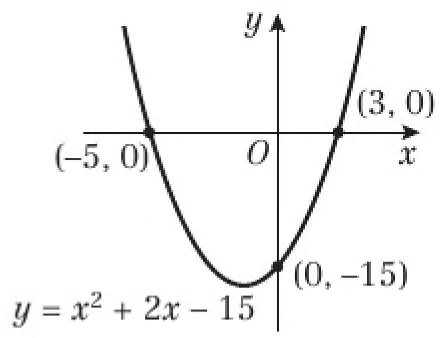
8 a $(3^x - 27)(3^x - 1)$ b 0 and 3

Exercise 2F

1 a 

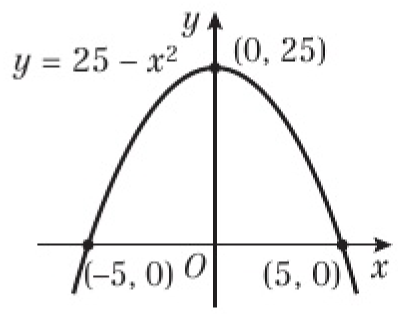
Turning point: $(3, -1)$
Line of symmetry: $x = 3$

b



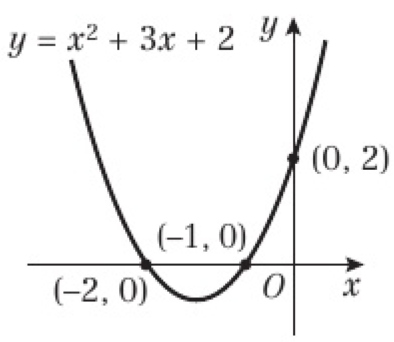
Turning point: $(-1, -16)$
Line of symmetry: $x = -1$

c



Turning point: $(0, 25)$
Line of symmetry: $x = 0$

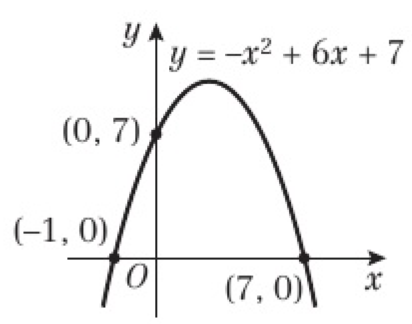
d



Turning point: $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

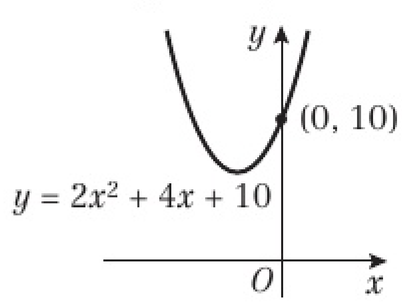
Line of symmetry: $x = -\frac{3}{2}$

e



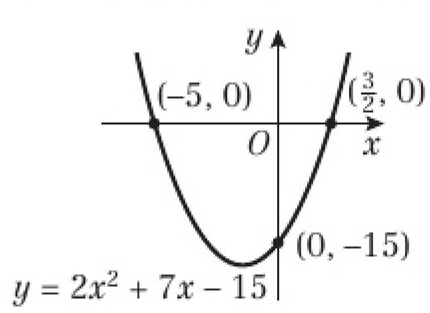
Turning point: $(3, 16)$
Line of symmetry: $x = 3$

f



Turning point: $(-1, 8)$
Line of symmetry: $x = -1$

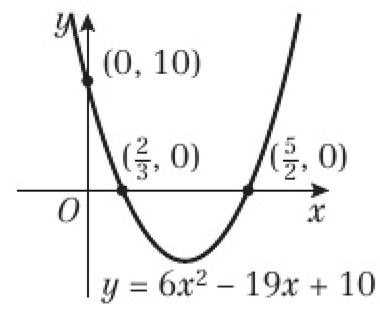
g



Turning point: $\left(-\frac{7}{4}, -\frac{169}{8}\right)$

Line of symmetry: $x = -\frac{7}{4}$

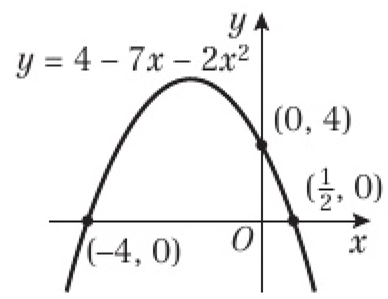
h



Turning point: $\left(\frac{19}{12}, -\frac{121}{24}\right)$

Line of symmetry: $x = \frac{19}{12}$

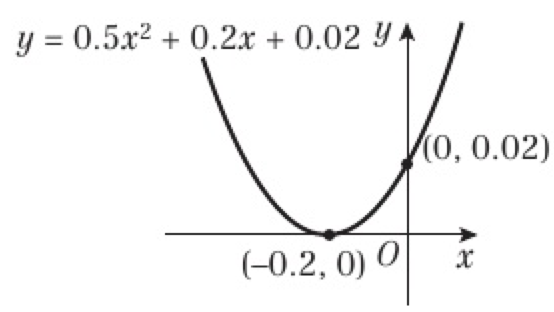
i



Turning point: $\left(-\frac{7}{4}, \frac{81}{8}\right)$

Line of symmetry: $x = -\frac{7}{4}$

j



Turning point: $(-0.2, 0)$
Line of symmetry: $x = -0.2$

2 **a** $a = 1, b = -8, c = 15$
b $a = -1, b = 3, c = 10$
c $a = 2, b = 0, c = -18$
d $a = \frac{1}{4}, b = -\frac{3}{4}, c = -1$

3 $a = 3, b = -30, c = 72$

Exercise 2G

1 **a** **i** 52 **ii** –23 **iii** 37 **iv** 0 **v** –44
b **i** $h(x)$ **ii** $f(x)$ **iii** $k(x)$ **iv** $j(x)$ **v** $g(x)$

2 $k < 9$

3 $t = \frac{9}{8}$

4 $s = 4$

5 $k > \frac{4}{3}$

6 **a** $p = 6$ **b** $x = -9$

7 **a** $k^2 + 16$
b k^2 is always positive so $k^2 + 16 > 0$

Challenge

a Need $b^2 > 4ac$. If $a, c > 0$ or $a, c < 0$, choose b such that $b > \sqrt{4ac}$. If $a > 0$ and $c < 0$ (or vice versa), then $4ac < 0$, so $4ac < b^2$ for all b.

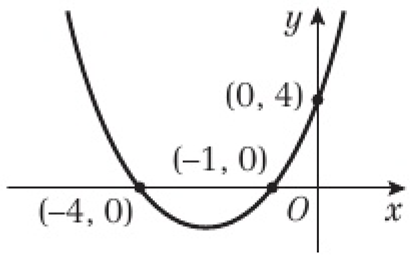
b Not if one of a or c are negative as this would require b to be the square root of a negative number. Possible if both negative or both positive.

Chapter review 2

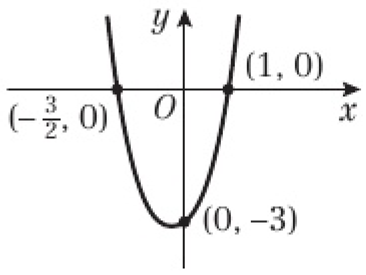
1 **a** $y = -1$ or -2 **b** $x = \frac{2}{3}$ or -5

c $x = -\frac{1}{5}$ or 3 **d** $x = \frac{5 \pm \sqrt{7}}{2}$

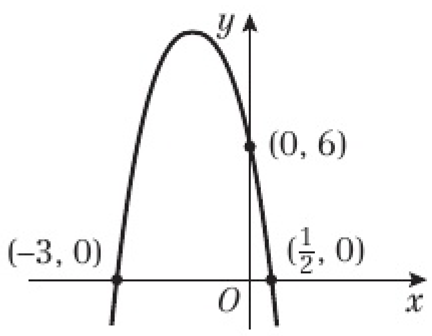
2 **a**



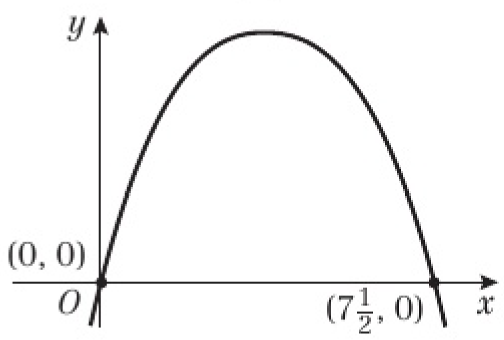
b



c



d



3 **a** $k = 1$ **b** $x = 3$ and $x = -2$

4 **a** $k = 0.0902$ or $k = -11.1$

b $t = 2.28$ or $t = 0.219$

c $x = -2.30$ or $x = 1.30$

d $x = 0.839$ or $x = -0.239$

5 **a** $(x + 6)^2 - 45$; $p = 1$, $q = 6$, $r = -45$

b $5(x - 4)^2 - 67$; $p = 5$, $q = -4$, $r = -67$

c $-2(x - 2)^2 + 8$; $p = -2$, $q = -2$, $r = 8$

d $2\left(x - \frac{1}{2}\right)^2 - \frac{3}{2}$; $p = 2$, $q = -\frac{1}{2}$, $r = -\frac{3}{2}$

6 $k = \frac{1}{5}$

7 **a** $p = 3$, $q = 2$, $r = -7$ **b** $-2 \pm \sqrt{\frac{7}{3}}$

8 **a** $f(x) = (2^x - 16)(2^x - 4)$ **b** 4 and 2

9 $1 \pm \sqrt{13}$

10 $x = -5$ or $x = 4$

11 **a** 10 m **b** 1.28 s

c $h(t) = 10.625 - 10(t - 0.25)^2$

$A = 10.625$, $B = 10$, $C = 0.25$

d 10.625 m at 0.25 s

12 **a** $16k^2 + 4$

b $k^2 \geqslant 0$ for all k, so $16k^2 + 4 > 0$

c When $k = 0$, $f(x) = 2x + 1$;

this is a linear function with only one root

13 1, –1, 2 and –2

Challenge

a $\frac{b + c}{b} = \frac{b}{c}$

$b^2 - cb - c^2 = 0$

Using quadratic formula: $b = \frac{c + \sqrt{5c^2}}{2}$

So $b : c$ is $\frac{c + \sqrt{5c^2}}{2} : c$

Dividing by c: $\frac{1 + \sqrt{5}}{2} : 1$

b Let $x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$

So $x = \sqrt{1 + x} \Rightarrow x^2 - x - 1 = 0$

Using quadratic formula: $x = \frac{1 + \sqrt{5}}{2}$

CHAPTER 3

Prior knowledge check

1 **a** $A \cap B = \{1, 2, 4\}$ **b** $(A \cup B)' = \{7, 9, 11, 13\}$

2 **a** $5\sqrt{3}$ **b** $\sqrt{5} + 2\sqrt{2}$

3 **a** graph **ii** **b** graph **iii** **c** graph **i**

Exercise 3A

1 **a** $x = 4, y = 2$ **b** $x = 1, y = 3$

c $x = 2, y = -2$ **d** $x = 4\frac{1}{2}, y = -3$

e $x = -\frac{2}{3}, y = 2$ **f** $x = 3, y = 3$

2 **a** $x = 5, y = 2$ **b** $x = 5\frac{1}{2}, y = -6$

c $x = 1, y = -4$ **d** $x = 1\frac{3}{4}, y = \frac{1}{4}$

3 **a** $x = -1, y = 1$ **b** $x = 4, y = -4$

c $x = 0.5, y = -2.5$

4 **a** $3x + ky = 8$ (1); $x - 2ky = 5$ (2)

(1) × 2: $6x + 2ky = 16$ (3)

(2) + (3) $7x = 21$ so $x = 3$

b –2

5 $p = 3, q = 1$

Exercise 3B

1 **a** $x = 5, y = 6$ or $x = 6, y = 5$

b $x = 0, y = 1$ or $x = \frac{4}{5}, y = -\frac{3}{5}$

c $x = -1, y = -3$ or $x = 1, y = 3$

d $a = 1, b = 5$ or $a = 3, b = -1$

e $u = 1\frac{1}{2}, v = 4$ or $u = 2, v = 3$

f $x = -1\frac{1}{2}, y = 5\frac{3}{4}$ or $x = 3, y = -1$

2 **a** $x = 3, y = \frac{1}{2}$ or $x = 6\frac{1}{3}, y = -2\frac{5}{6}$

b $x = 4\frac{1}{2}, y = 4\frac{1}{2}$ or $x = 6, y = 3$

c $x = -19, y = -15$ or $x = 6, y = 5$

3 **a** $x = 3 + \sqrt{13}, y = -3 + \sqrt{13}$ or $x = 3 - \sqrt{13}$,

$y = -3 - \sqrt{13}$

b $x = 2 - 3\sqrt{5}, y = 3 + 2\sqrt{5}$ or $x = 2 + 3\sqrt{5}, y = 3 - 2\sqrt{5}$

4 $x = -5, y = 8$ or $x = 2, y = 1$

5 **a** $3x^2 + x(2 - 4x) + 11 = 0$

$3x^2 + 2x - 4x^2 + 11 = 0$

$x^2 - 2x - 11 = 0$

b $x = 1 + 2\sqrt{3}, y = -2 - 8\sqrt{3}$

$x = 1 - 2\sqrt{3}, y = -2 + 8\sqrt{3}$

6 **a** $k = 3, p = -2$

b $x = -6, y = -23$

Challenge

$y = x + k$

$x^2 + (x + k)^2 = 4$

$x^2 + x^2 + 2kx + k^2 - 4 = 0$

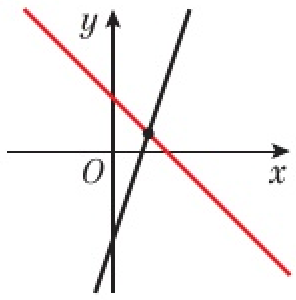
$2x^2 + 2kx + k^2 - 4 = 0$ for one solution $b^2 - 4ac = 0$

Online Worked solutions are available in SolutionBank.

$4k^2 - 4 \times 2(k^2 - 4) = 0$
$4k^2 - 8k^2 + 32 = 0 \quad 4k^2 = 32 \quad k^2 = 8 \quad k = \pm 2\sqrt{2}$

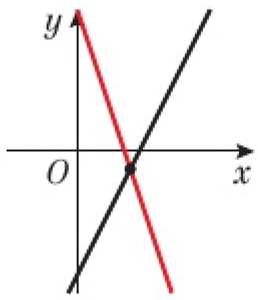
Exercise 3C

1 a i



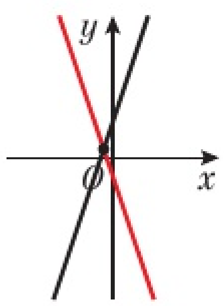
ii (2, 1)

b i



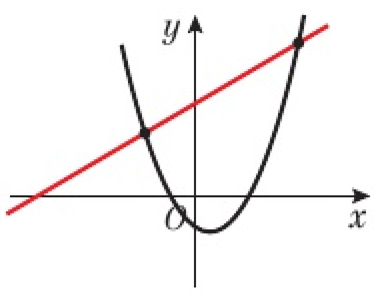
ii (3, –1)

c i



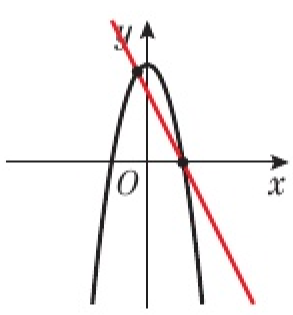
ii (–0.5, 0.5)

2 a



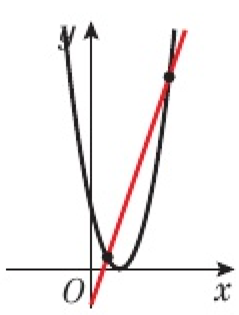
b (3.5, 9) and (–1.5, 4)

3 a



b (–1, 8) and (3, 0)

4 a



b (6, 16) and (1, 1)

5 (–11, –15) and (3, –1)

6 $(-1\frac{1}{6}, -4\frac{1}{2})$ and (2, 5)

7 a 2 points **b** 1 point **c** 0 points

8 a $y = 2x - 1$
$x^2 + 4k(2x - 1) + 5k = 0$
$x^2 + 8kx - 4k + 5k = 0 \quad x^2 + 8kx + k = 0$

b $k = \frac{1}{16}$ **c** $x = -\frac{1}{4}, y = -\frac{3}{2}$

Exercise 3D

1 a $x < 4$ **b** $x \geqslant 7$ **c** $x > 2\frac{1}{2}$ **d** $x \leqslant -3$
e $x < 11$ **f** $x < 2\frac{3}{5}$ **g** $x > -12$ **h** $x < 1$
i $x \leqslant 8$ **j** $x > 1\frac{1}{7}$

2 a $x \geqslant 3$ **b** $x < 1$ **c** $x \leqslant -3\frac{1}{4}$ **d** $x < 18$
e $x > 3$ **f** $x \geqslant 4\frac{2}{5}$ **g** $x < 4$ **h** $x > -7$
i $x \leqslant -\frac{1}{2}$ **j** $x \geqslant \frac{3}{4}$ **k** $x \geqslant -\frac{10}{3}$ **l** $x \geqslant \frac{9}{11}$

3 a $\{x: x > 2\frac{1}{2}\}$ **b** $\{x: 2 < x < 4\}$
c $\{x: 2\frac{1}{2} < x < 3\}$ **d** No values
e $x = 4$ **f** $\{x: x < 1.2\} \cup \{x: x > 2.2\}$
g $\left\{x: x \leqslant -\frac{2}{3}\right\} \cup \left\{x: x \geqslant \frac{3}{2}\right\}$

Challenge

$p = -1, q = 4, r = 6$

Exercise 3E

1 a $3 < x < 8$ **b** $-4 < x < 3$
c $x < -2, x > 5$ **d** $x \leqslant -4, x \geqslant -3$
e $-\frac{1}{2} < x < 7$ **f** $x < -2, x > 2\frac{1}{2}$
g $\frac{1}{2} \leqslant x \leqslant 1\frac{1}{2}$ **h** $x < \frac{1}{3}, x > 2$
i $-3 < x < 3$ **j** $x < -2\frac{1}{2}, x > \frac{2}{3}$
k $x < 0, x > 5$ **l** $-1\frac{1}{2} \leqslant x \leqslant 0$

2 a $-5 < x < 2$ **b** $x < -1, x > 1$
c $\frac{1}{2} < x < 1$ **d** $-3 < x < \frac{1}{4}$

3 a $\{x: 2 < x < 4\}$ **b** $\{x: x > 3\}$
c $\{x: -\frac{1}{4} < x < 0\}$ **d** No values
e $\{x: -5 < x < -3\} \cup \{x: x > 4\}$
f $\{x: -1 < x < 1\} \cup \{x: 2 < x < 3\}$

4 a $x < 0$ or $x > 2$ **b** $x < 0$ or $x > 0.8$
c $x < -1$ or $x > 0$ **d** $x < 0$ or $x > 0.5$
e $x < -\frac{1}{5}$ or $x > \frac{1}{5}$ **f** $x \leqslant -\frac{2}{3}$ or $x \geqslant 3$

5 a $-2 < k < 6$ **b** $p \leqslant -8$ or $p \geqslant 0$

6 $\{x: x < -2\} \cup \{x: x > 7\}$

7 a $\{x: x < \frac{2}{3}\}$ **b** $\{x: -\frac{1}{2} < x < 3\}$
c $\{x: -\frac{1}{2} < x < \frac{2}{3}\}$

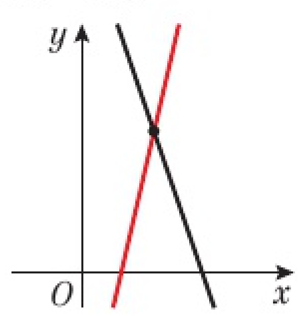
8 $x < 3$ or $x > 5.5$

9 No real roots $b^2 - 4ac < 0 \quad (-2k)^2 - 4 \times k \times 3 < 0$
$4k^2 - 12k = 0$ when $k = 0$ and $k = 3$
solution $0 \leqslant k < 3$
note when $k = 0$ equation gives $3 = 0$

Exercise 3F

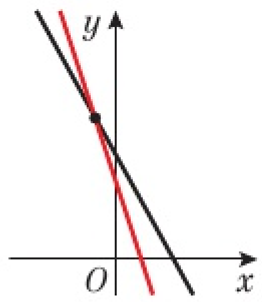
1 a $P(3.2, -1.8)$ **b** $x < 3.2$

2 a i



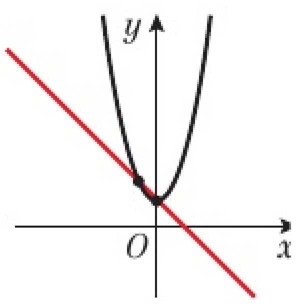
ii (4, 5) iii $x \leqslant 4$

b i



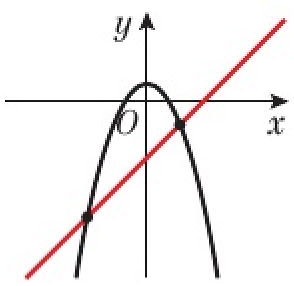
ii (–3, 23) iii $x \geqslant -3$

c i



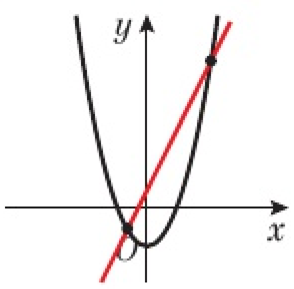
ii (–2, 9), (0, 5) iii $-2 \leqslant x \leqslant 0$

d i



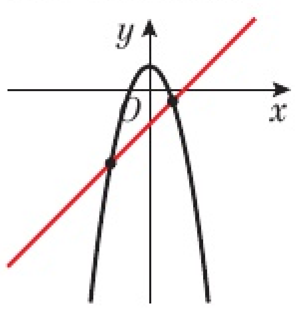
ii (–5, –22), (3, –6) iii $x \leqslant -5$ or $x \geqslant 3$

e i



ii (–2, –1), (9, 76) iii $-2 \leqslant x \leqslant 9$

f i



ii (–5, –18), (3, –2) iii $x \leqslant -5$ or $x \geqslant 3$

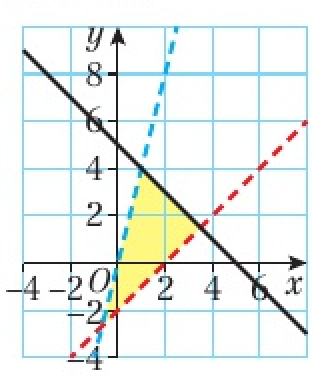
3 a $-1 < x < 2$ b $0.5 < x < 3$
c $x < 0.5$ or $x > 3$ d $x < 0$ or $x > 2$
e $1 < x < 3$ f $x < -1$ or $x > -0.75$

Challenge

a (–1.5, –3.75), (6, 0)
b $\{x: -1.5 < x < 6\}$

Exercise 3G

1



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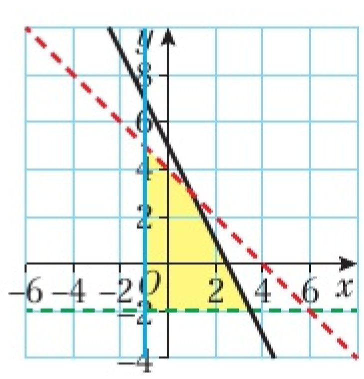

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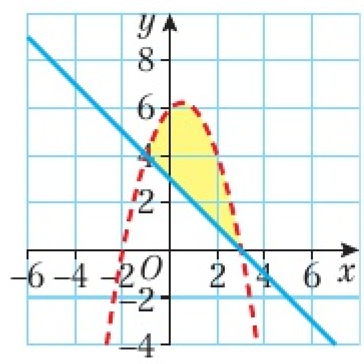

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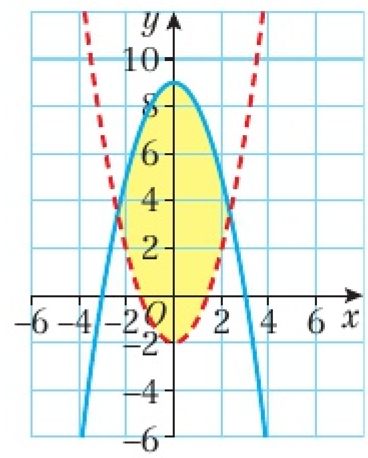

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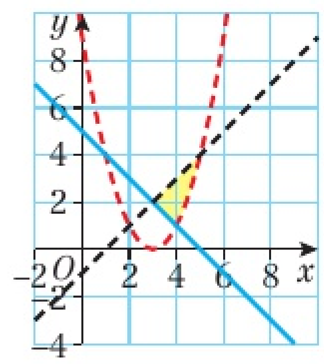
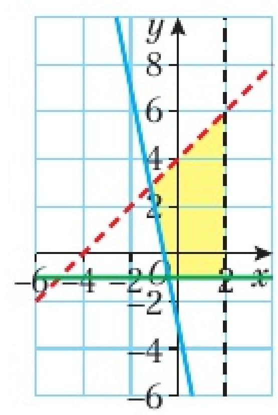


6 a (1, 6), (3, 4), (1, 2)
b $x \geqslant 1$, $y \leqslant 7 - x$, $y \geqslant x + 1$

7 $y < 2 - 5x - x^2$, $2x + y \geqslant 0$, $x + y \leqslant 4$

8 a



b $\left(-\frac{7}{6}, \frac{17}{6}\right)$, (2, 6), (2, –1), (–0.4, –1)
c (–0.4, –1) d $\frac{941}{60}$

Chapter review 3

1 a
$$4kx - 2y = 8$$
$$4kx + 3y = -2$$
$$-5y = 10$$
$$y = -2$$
b $x = \dfrac{1}{k}$

2 $x = -4$, $y = 3\frac{1}{2}$

3 a Substitute $x = 1 + 2y$ into $3xy - y^2 = 8$
b (3, 1) and $\left(-\frac{11}{5}, -\frac{8}{5}\right)$

4 a Substitute $y = 2 - x$ into $x^2 + xy - y^2 = 0$
b $x = 3 \pm \sqrt{6}$, $y = -1 \pm \sqrt{6}$

5 a $3^x = (3^2)^{y-1} = 3^{2y-2} \Rightarrow x = 2y - 2$
b $x = 4$, $y = 3$ and $x = -2\frac{2}{3}$, $y = -\frac{1}{3}$

6 $x = -1\frac{1}{2}$, $y = 2\frac{1}{4}$ and $x = 4$, $y = -\frac{1}{2}$

7 a $k = -2$ b (–1, 2)

8 a $\{x: x > 10\frac{1}{2}\}$ b $\{x: x < -2\} \cup \{x: x > 7\}$

9 $3 < x < 4$

10 a $x = -5$, $x = 4$ b $\{x: x < -5\} \cup \{x: x > 4\}$

11 a $x < 2\frac{1}{2}$ b $\frac{1}{2} < x < 5$
c $0 < x < 4$ d $\frac{1}{2} < x < 2\frac{1}{2}$

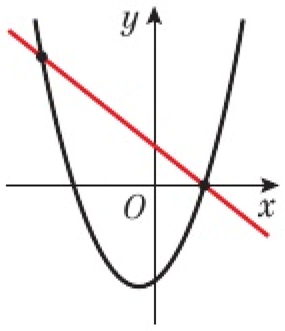
12 $1 \leqslant x \leqslant 8$

13 $k \leqslant 3\frac{1}{5}$

14 $b^2 < 4ac$ so $16k^2 < -40k$

$8k(2k + 5) < 0$ so $-\frac{5}{2} < k < 0$

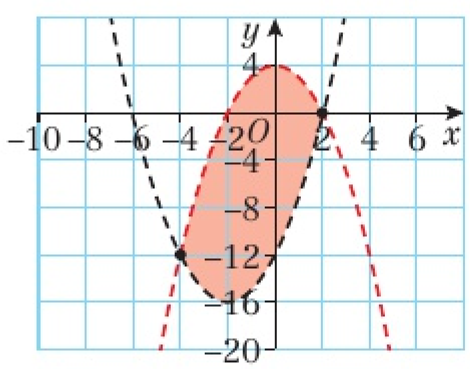
15 a



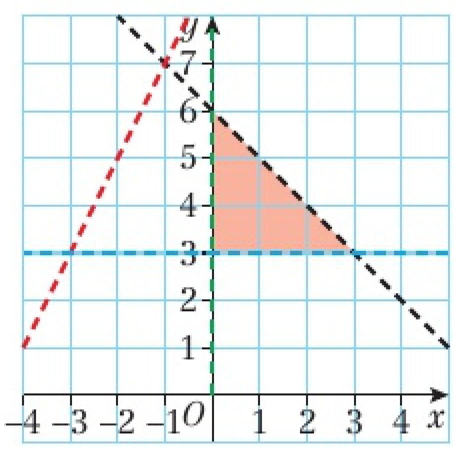
b $(-7, 20), (3, 0)$ **c** $x < -7, x > 3$

16 $\frac{1}{4}(-1 - \sqrt{185}) < x < \frac{1}{4}(-1 + \sqrt{185})$

17



18 a



b $\frac{9}{2}$

Challenge

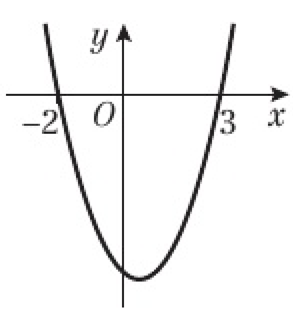
1 $0 < x < 1.6$ **2** $-2 < k < 7$

CHAPTER 4

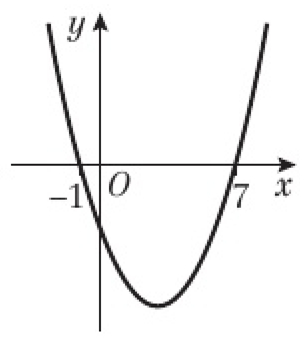
Prior knowledge check

1 a $(x + 5)(x + 1)$ **b** $(x - 3)(x - 1)$

2 a



b

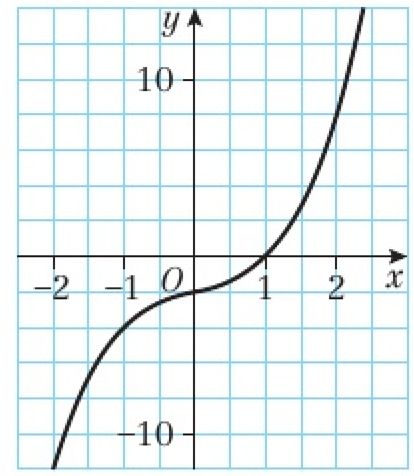


3 a

x	−2	−1.5	−1	−0.5	0
y	−12	−6.875	−4	−2.625	−2

x	0.5	1	1.5	2
y	−1.375	0	2.875	8

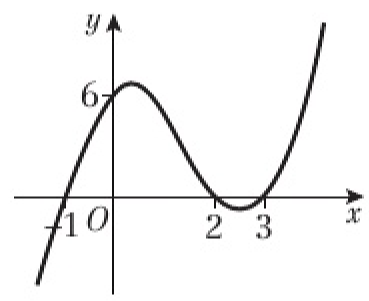
b



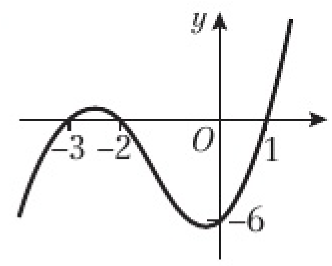
4 a $x = 2, y = 4$ **b** $x = 1, y = 1$

Exercise 4A

1 a



b



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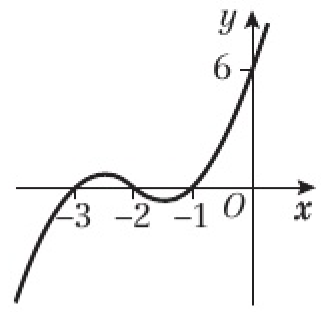

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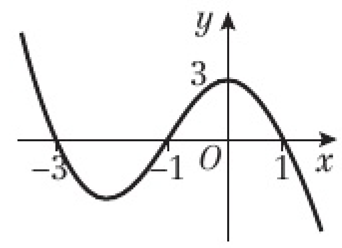

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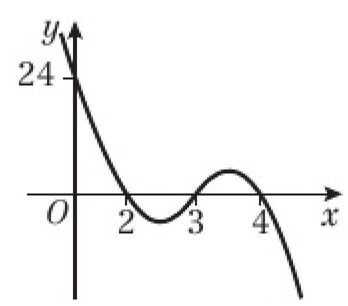

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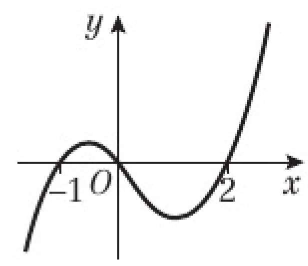

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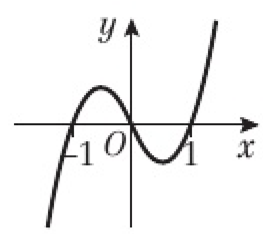

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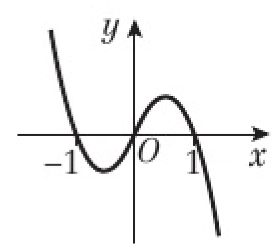

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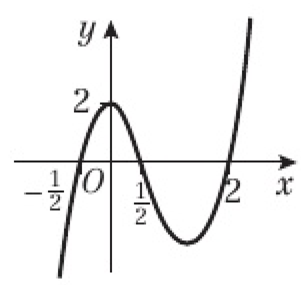

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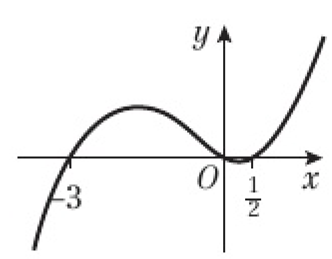
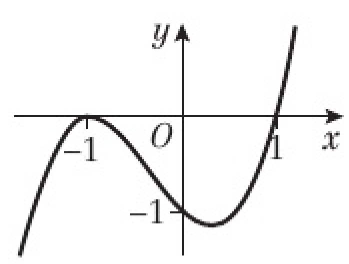
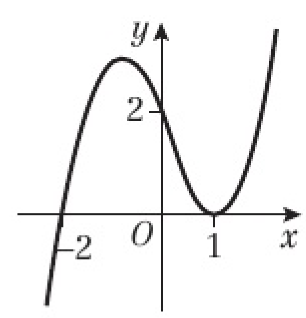


2 a



b



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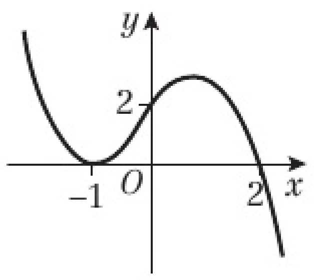
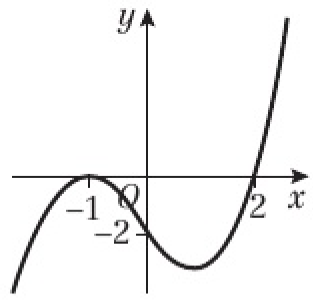
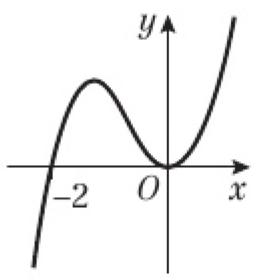


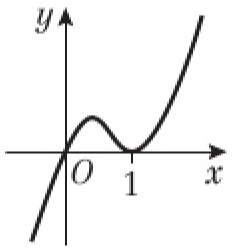
d



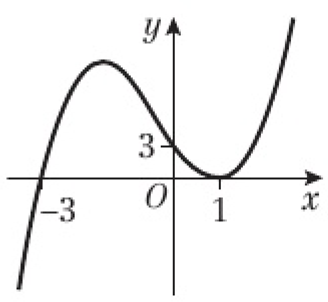
e



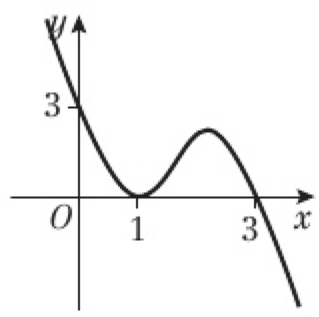
f



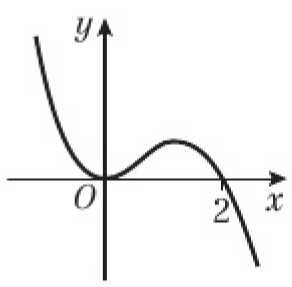
g



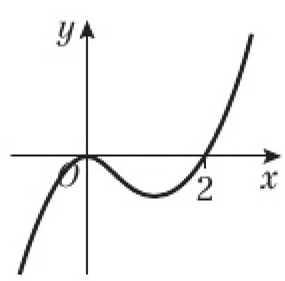
h



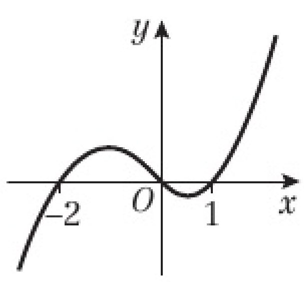
i



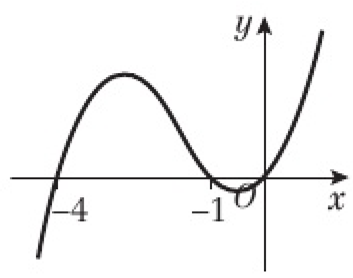
j



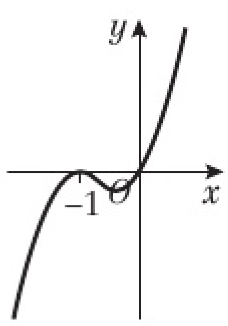
3 a $y = x(x + 2)(x - 1)$



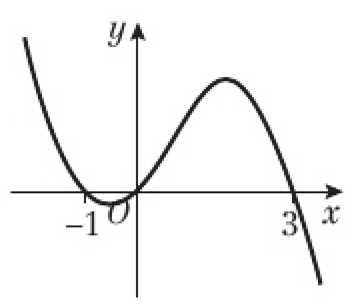
b $y = x(x + 4)(x + 1)$



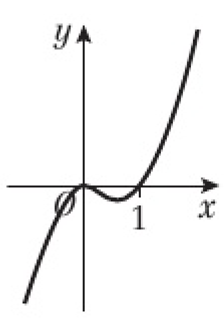
c $y = x(x + 1)^2$



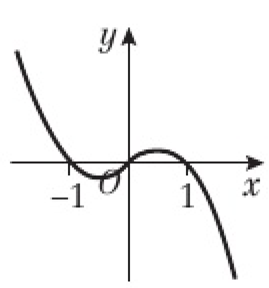
d $y = x(x + 1)(3 - x)$



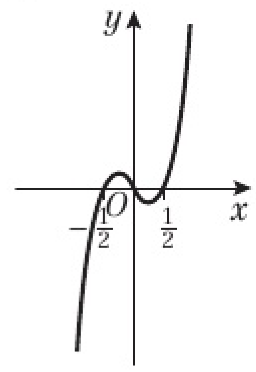
e $y = x^2(x - 1)$



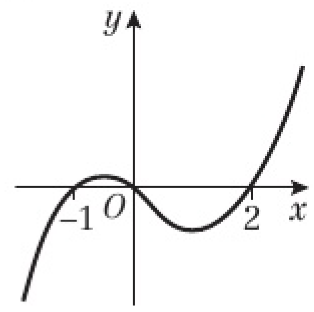
f $y = x(1 - x)(1 + x)$



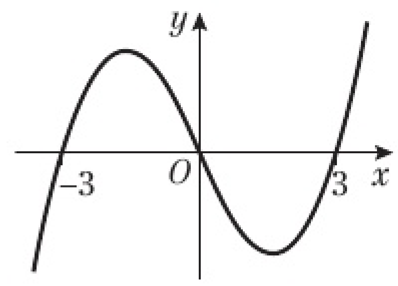
g $y = 3x(2x - 1)(2x + 1)$



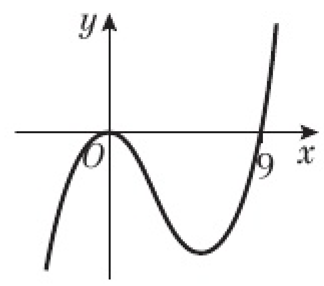
h $y = x(x + 1)(x - 2)$



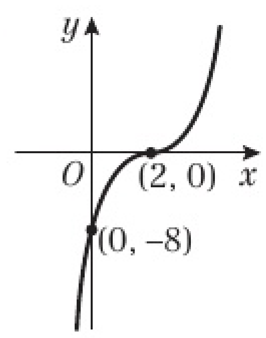
i $y = x(x - 3)(x + 3)$



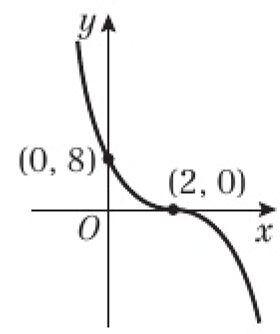
j $y = x^2(x - 9)$



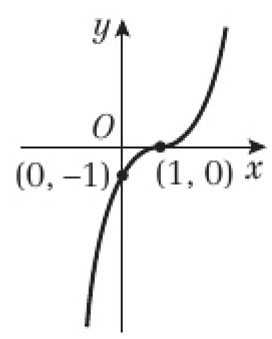
4 a



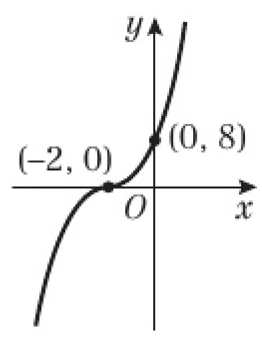
b



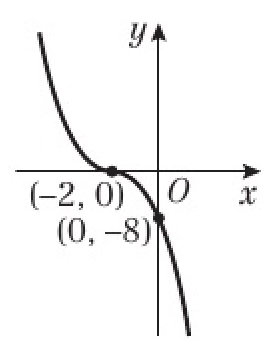
c



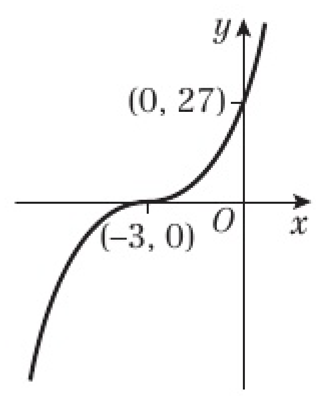
d



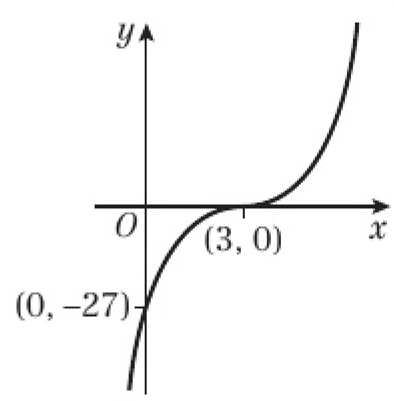
e



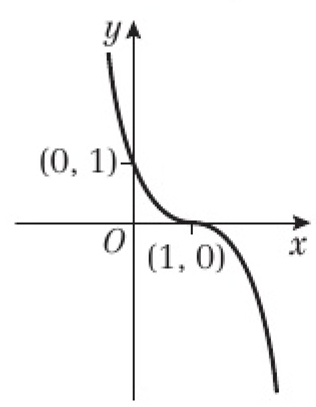
f



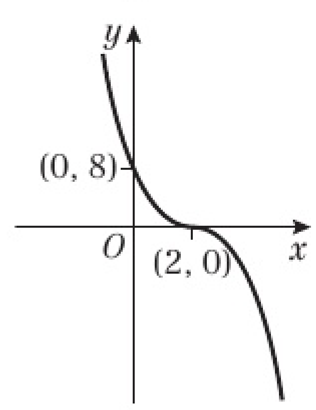
g



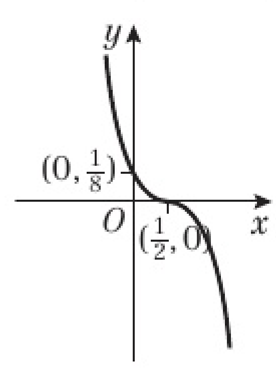
h



i



j



Online Worked solutions are available in SolutionBank.

5 **a** $b = 4, c = 1, d = -6$

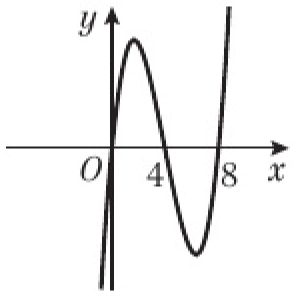
b $(0, -6)$

6 $a = \frac{1}{3}, b = -\frac{4}{3}, c = \frac{1}{3}, d = 2$

7 **a** $x(x^2 - 12x + 32)$

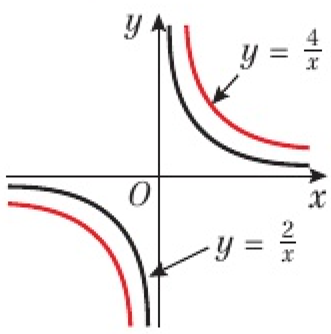
b $x(x - 8)(x - 4)$

c

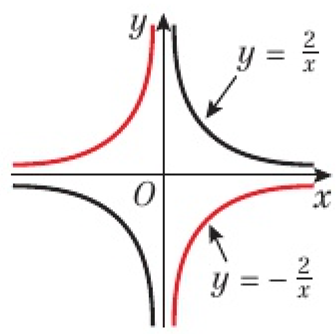


Exercise 4B

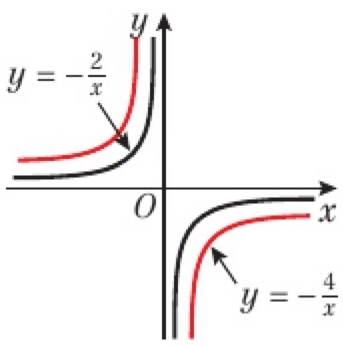
1 **a**



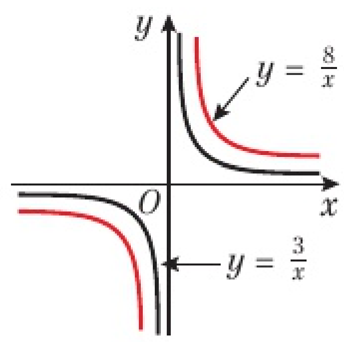
b



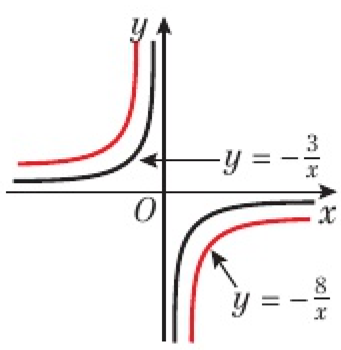
c



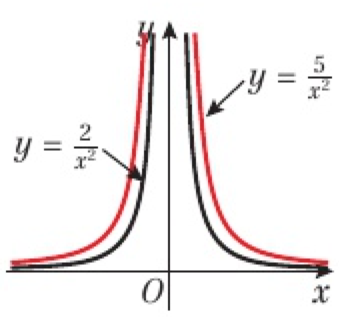
d



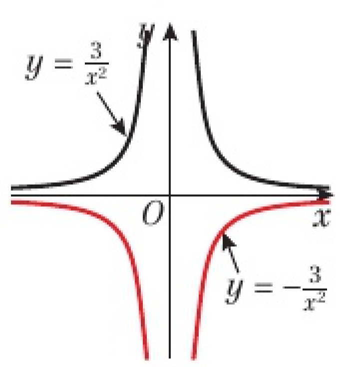
e



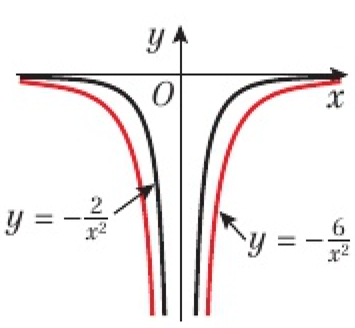
2 **a**



b

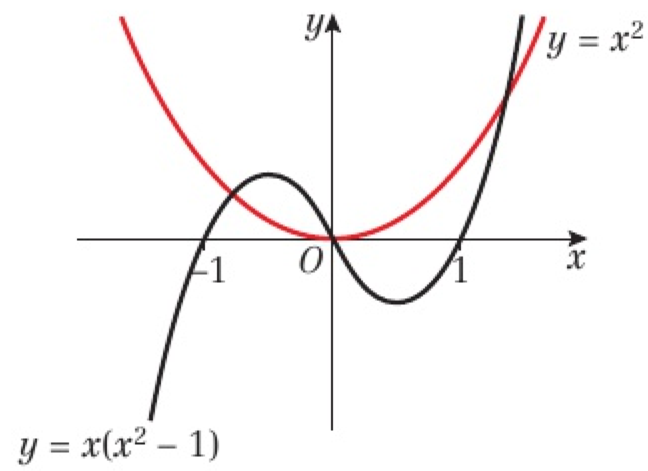


c



Exercise 4C

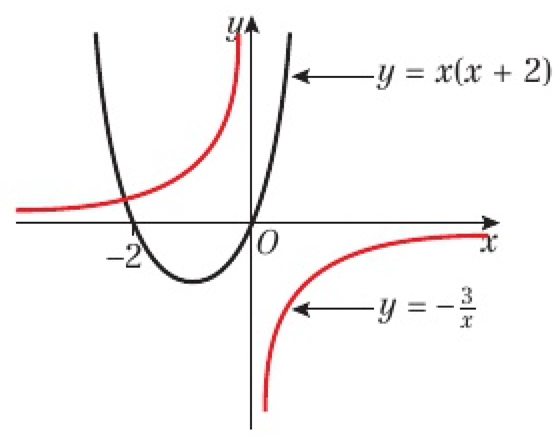
1 **a** **i**



ii 3

iii $x^2 = x(x^2 - 1)$

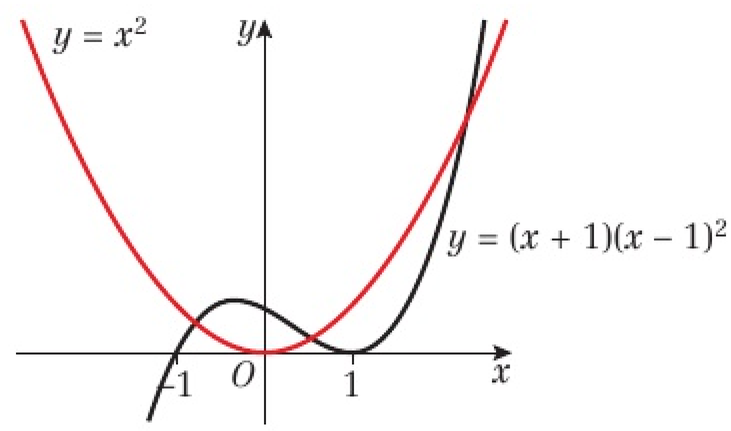
b **i**



ii 1

iii $x(x + 2) = -\dfrac{3}{x}$

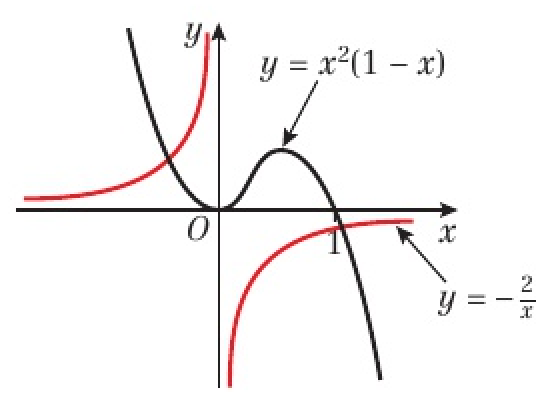
c **i**



ii 3

iii $x^2 = (x + 1)(x - 1)^2$

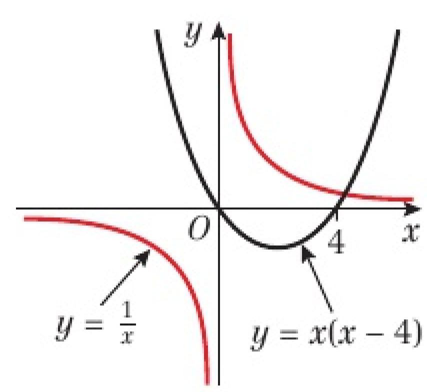
d **i**



ii 2

iii $x^2(1 - x) = -\dfrac{2}{x}$

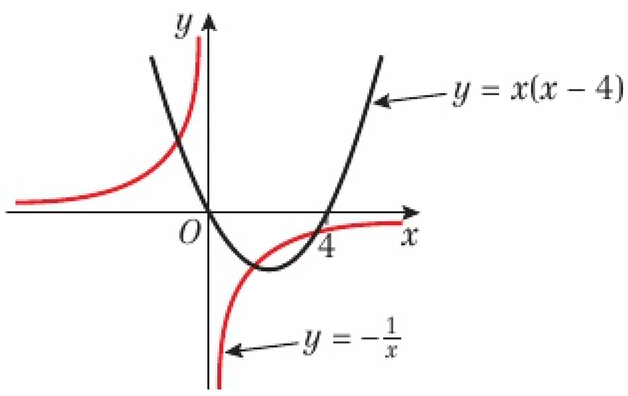
e **i**



ii 1

iii $x(x-4)=\frac{1}{x}$

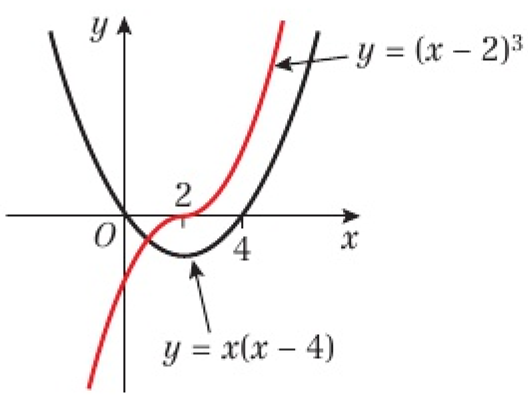
f i



ii 3

iii $x(x-4)=-\frac{1}{x}$

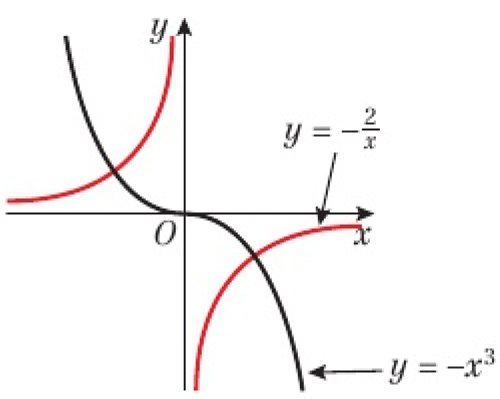
g i



ii 1

iii $x(x-4)=(x-2)^3$

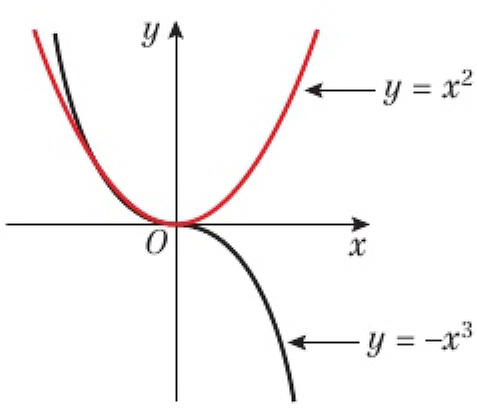
h i



ii 2

iii $-x^3=-\frac{2}{x}$

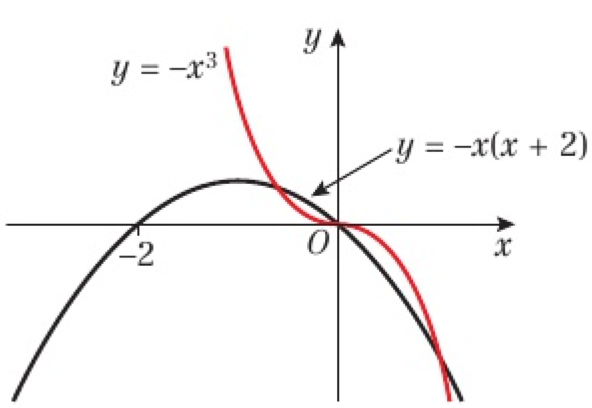
i i



ii 2

iii $-x^3=x^2$

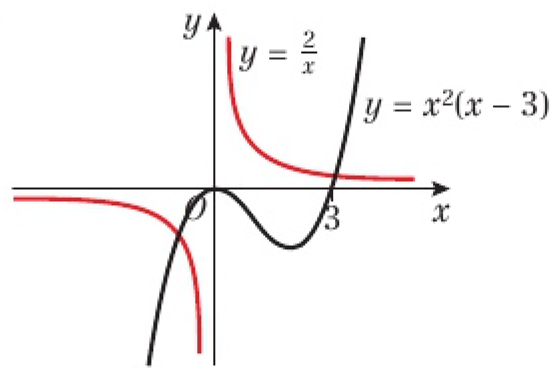
j i



ii 3

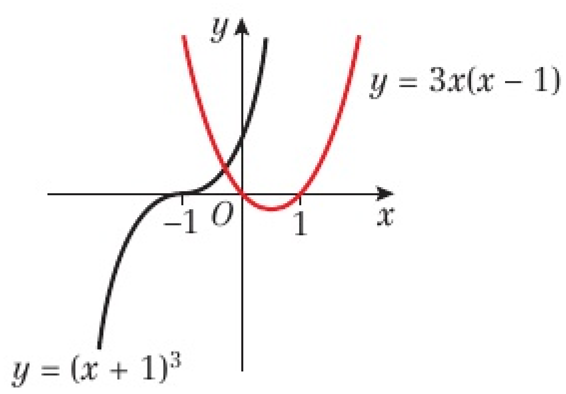
iii $-x^3=-x(x+2)$

2 a



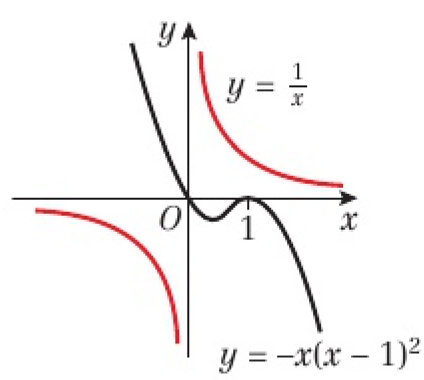
b Only 2 intersections

3 a



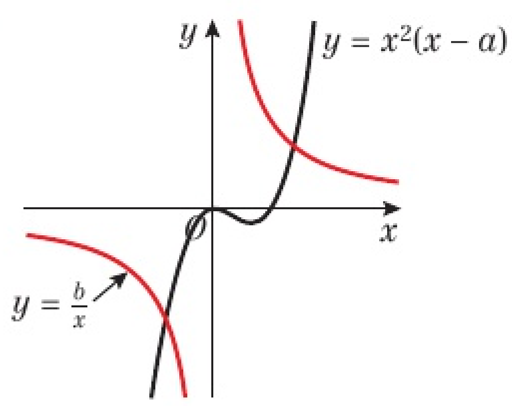
b Only 1 intersection

4 a



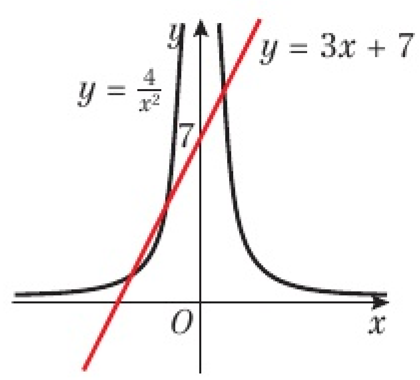
b Graphs do not intersect

5 a



b 2; the graphs cross in two places so there are two solutions.

6 a

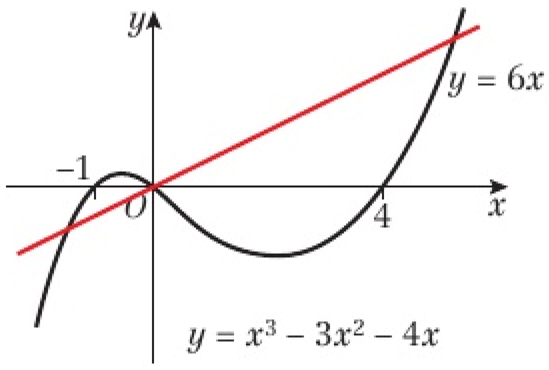


b 3

c Expand brackets and rearrange.

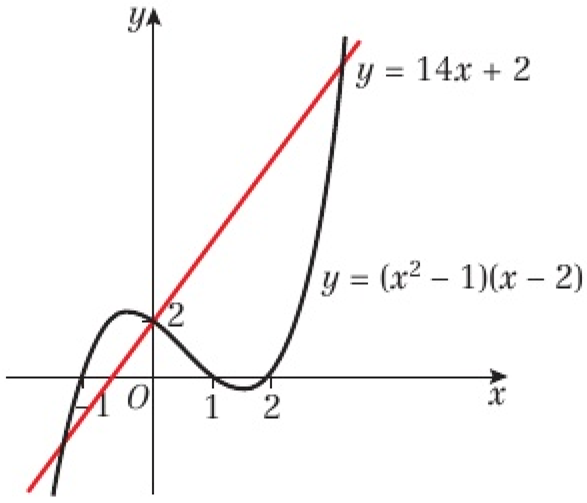
d $(-2, 1)$, $(-1, 4)$, $(\frac{2}{3}, 9)$

7 a



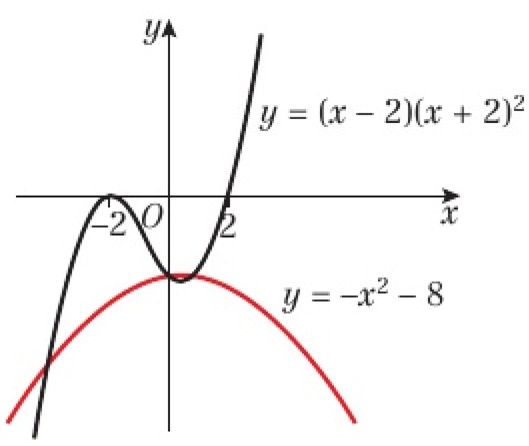
b (0, 0); (–2, –12); (5, 30)

8 a



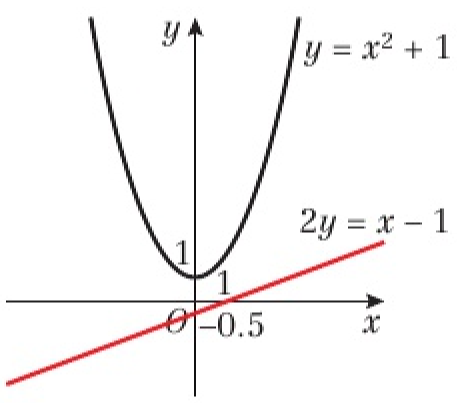
b (0, 2); (–3, –40); (5, 72)

9 a



b (0, –8); (1, –9); (–4, –24)

10 a

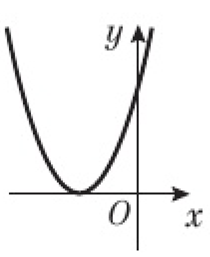


b Graphs do not intersect.

c $a < -\frac{7}{16}$

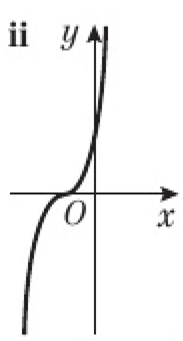
Exercise 4D

1 a i



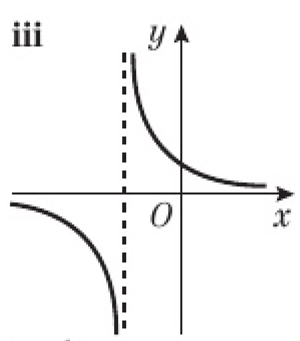
(–2, 0), (0, 4)

ii



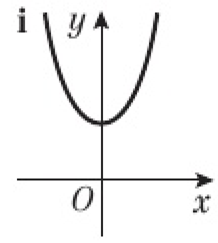
(–2, 0), (0, 8)

iii



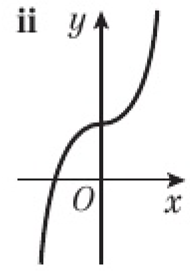
$(0, \frac{1}{2})$, $x = -2$, $y = 0$

b i



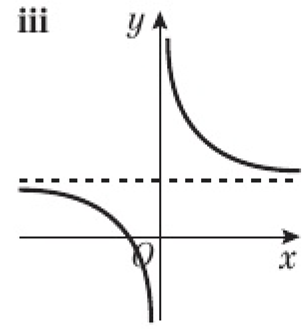
(0, 2)

ii



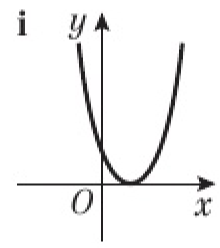
$(-\sqrt[3]{2}, 0)$, (0, 2)

iii



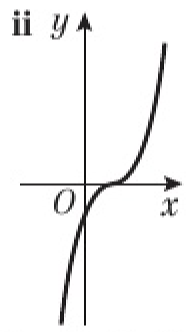
$(-\frac{1}{2}, 0)$, $y = 2$, $x = 0$

c i



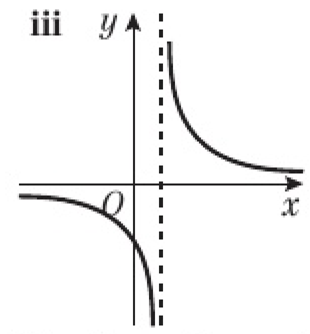
(0, 1), (1, 0)

ii



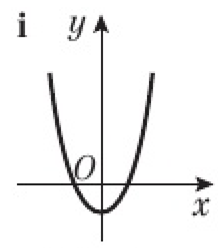
(0, –1), (1, 0)

iii



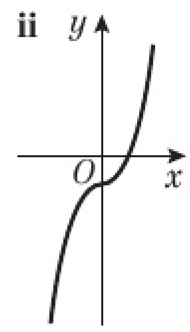
(0, –1), $x = 1$, $y = 0$

d i



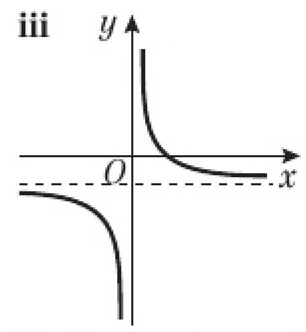
(–1, 0),
(0, –1),
(1, 0)

ii



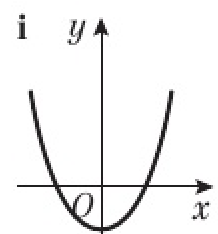
(0, –1), (1, 0)

iii



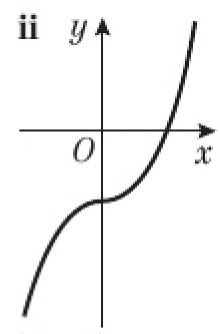
(1, 0), $y = -1$, $x = 0$

e i



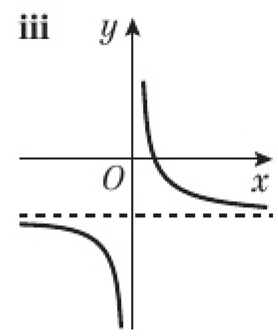
$(-\sqrt{3}, 0)$,
(0, –3),
$(\sqrt{3}, 0)$

ii



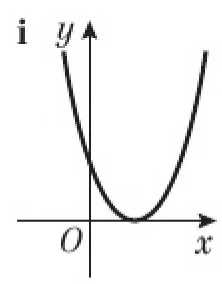
(0, –3),
$(\sqrt[3]{3}, 0)$

iii



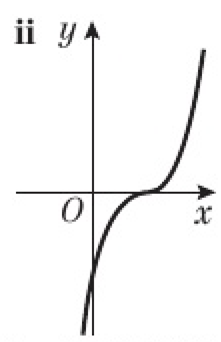
$(\frac{1}{3}, 0)$, $y = -3$, $x = 0$

f i



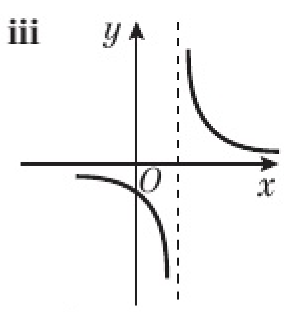
(0, 9), (3, 0)

ii



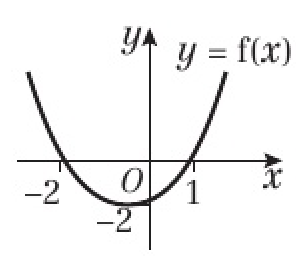
(0, –27), (3, 0)

iii

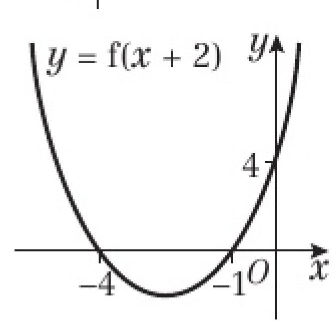


$(0, -\frac{1}{3})$, $x = 3$, $y = 0$

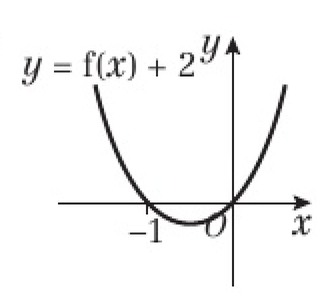
2 a



b i

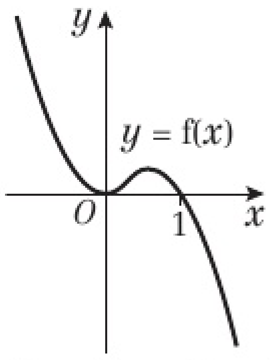


ii

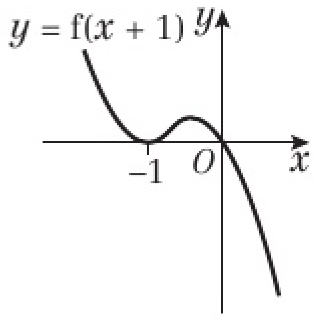


c f(x + 2) = (x + 1)(x + 4); (0, 4)
f(x) + 2 = (x − 1)(x + 2) + 2; (0, 0)

3 a

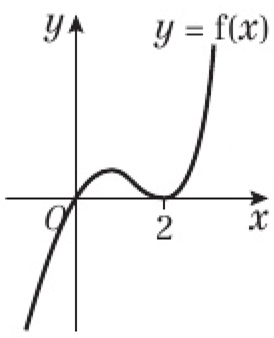


b

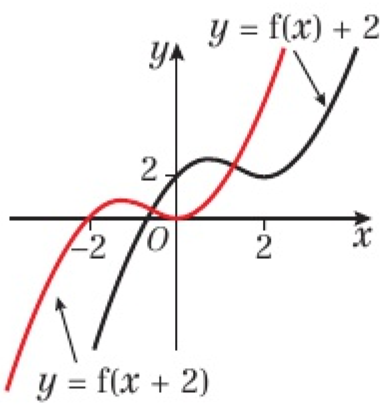


c f(x + 1) = −x(x + 1)²; (0, 0)

4 a

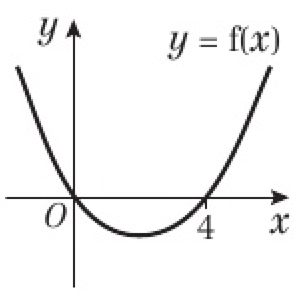


b

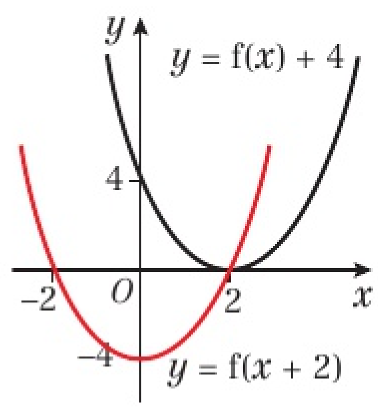


c f(x + 2) = (x + 2)x²; (0, 0); (−2, 0)

5 a



b

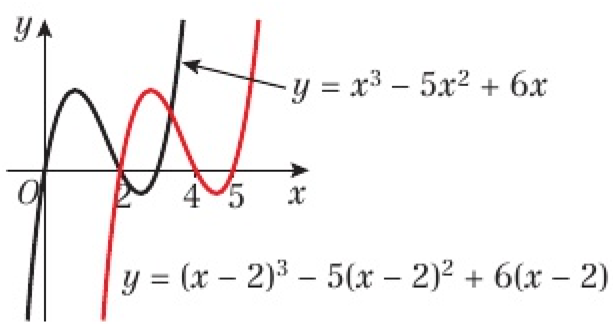


c f(x + 2) = (x + 2)(x − 2); (2, 0); (−2, 0); (0, −4)
f(x) + 4 = (x − 2)²; (2, 0); (0, 4)

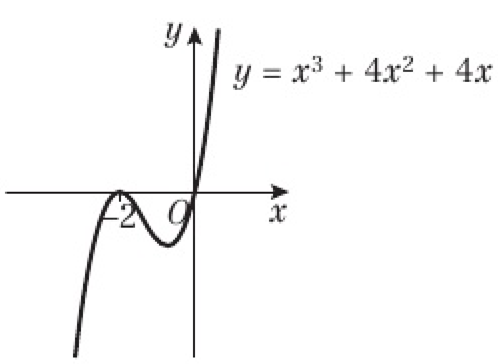
6 a (6, −1) **b** (4, 2)

7 $y = \frac{1}{x-4}$

8



9 a



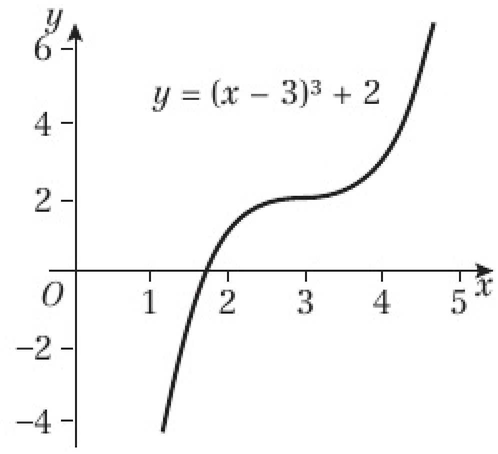
b −1 or 1

Challenge

1 (3, 2)

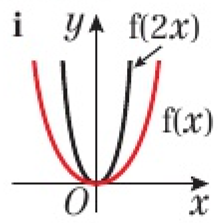
2 a (−7, −12)

b f(x − 2) + 1

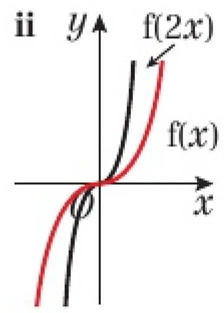


Exercise 4E

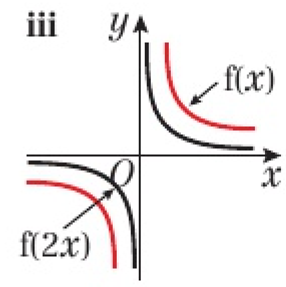
1 a i



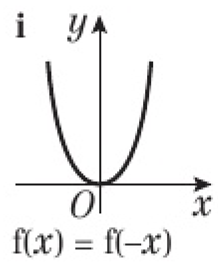
ii



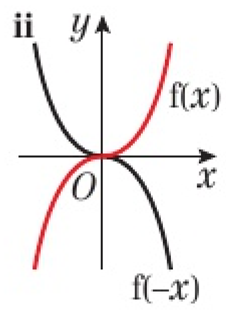
iii



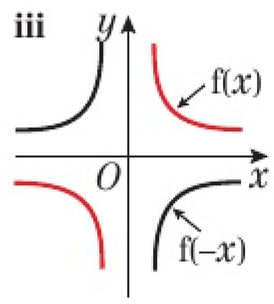
b i



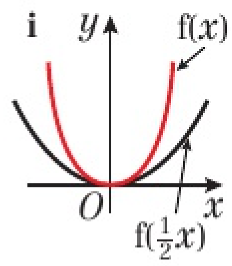
ii



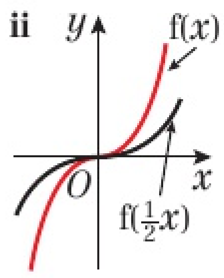
iii



c i



ii



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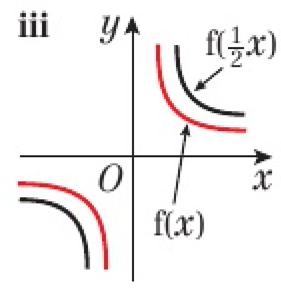
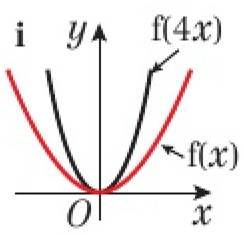


d i



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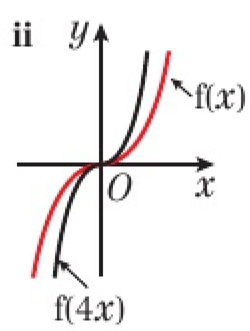

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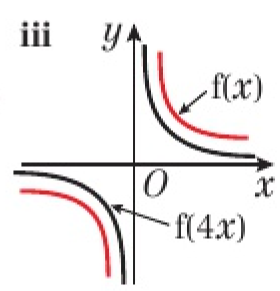
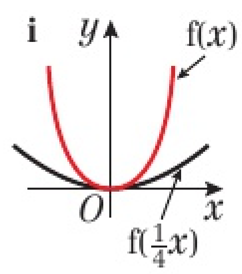


e i



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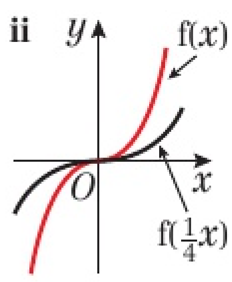

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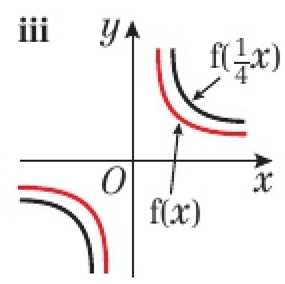
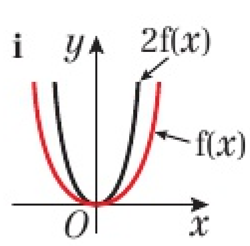


f i



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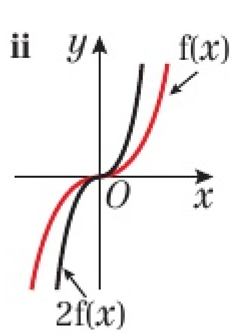

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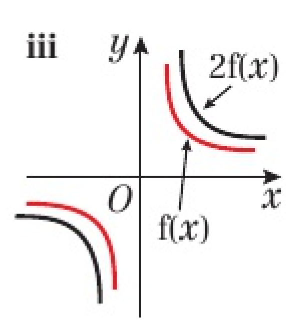
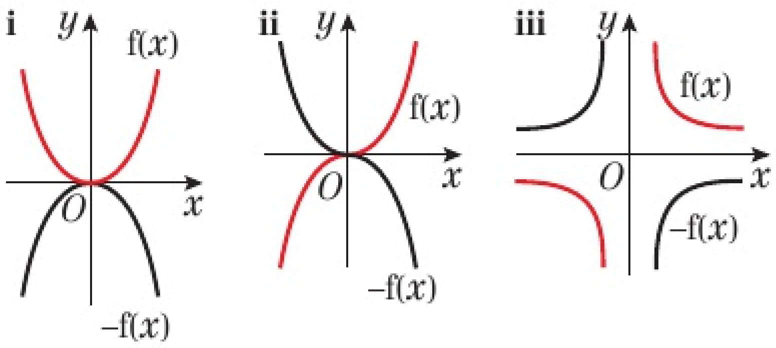
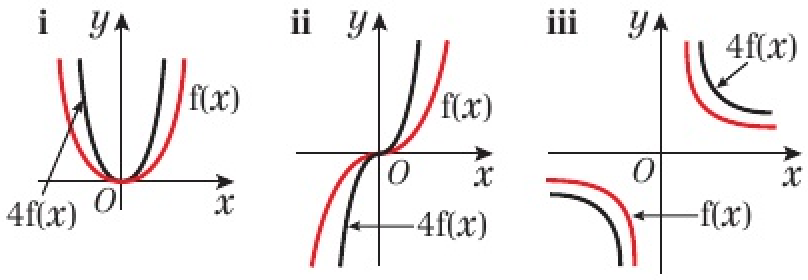
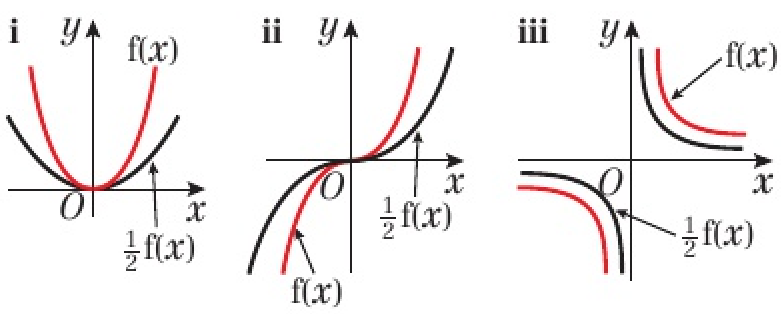


Online Worked solutions are available in SolutionBank.

g


h


i


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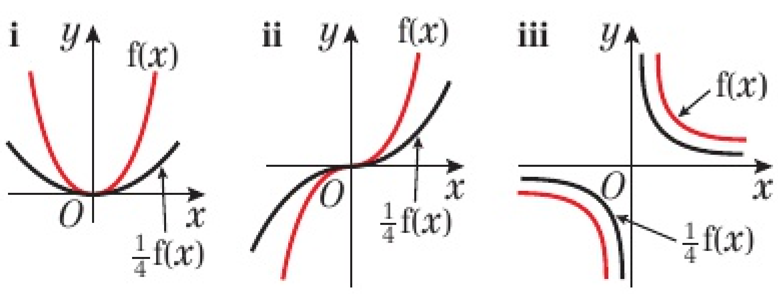
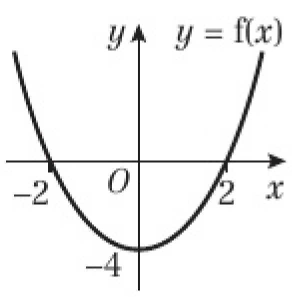


2 a


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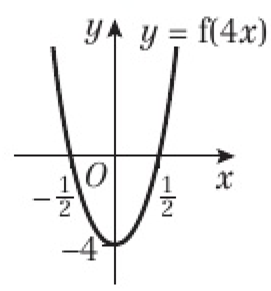
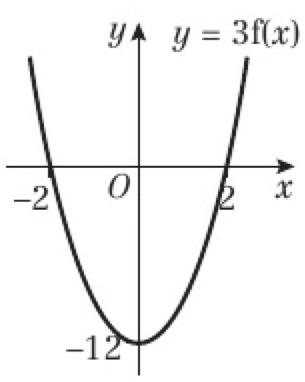
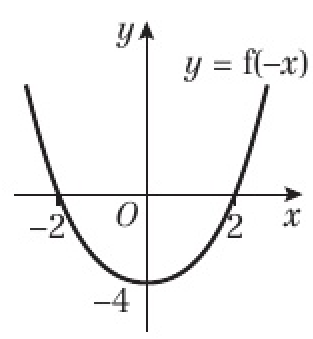
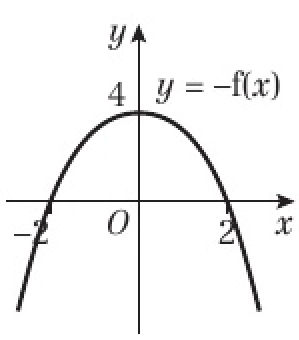
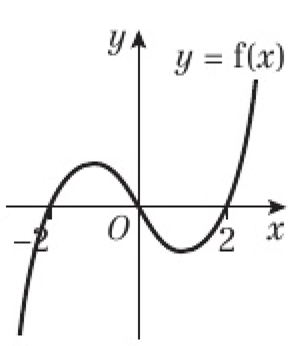








3 a


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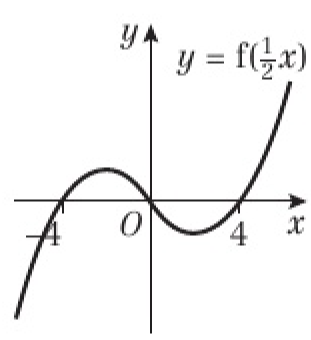

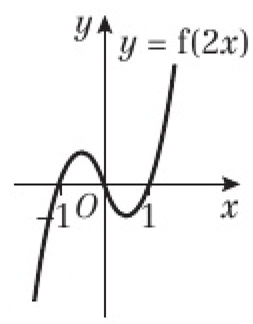

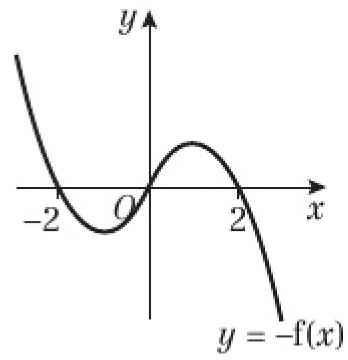

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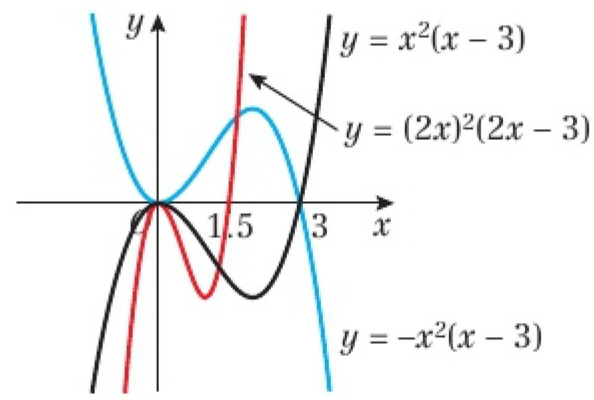

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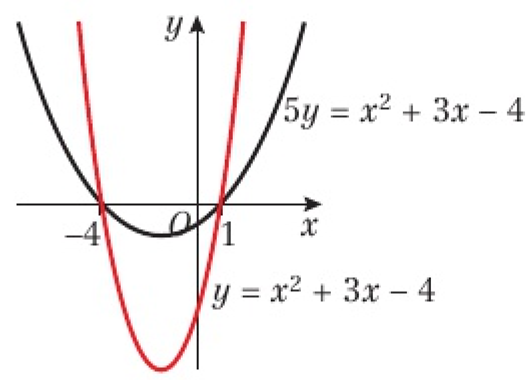

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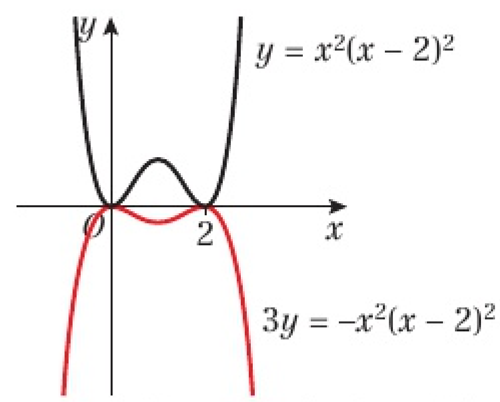
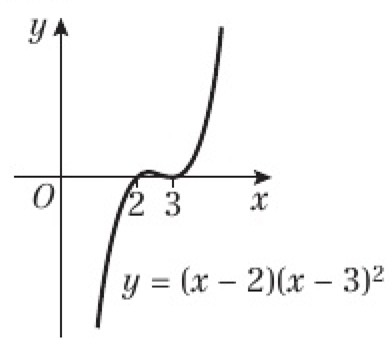


7 **a** $(1, -3)$ **b** $(2, -12)$

8 $(-4, 8)$

9 **a**


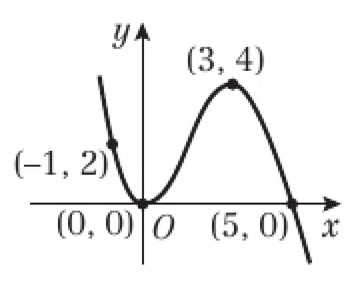
b 2 and 3

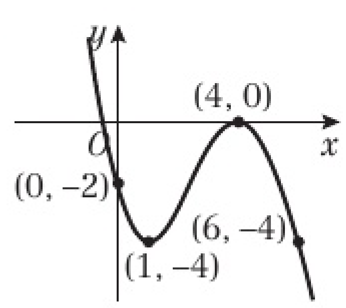
Challenge

1 $(2, -2)$

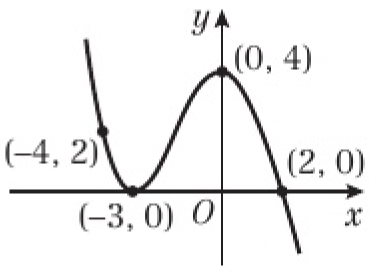
2 $\frac{1}{4}f(\frac{1}{2}x)$

Exercise 4F

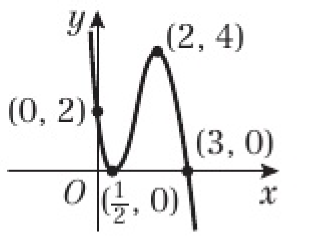
1 **a**


b


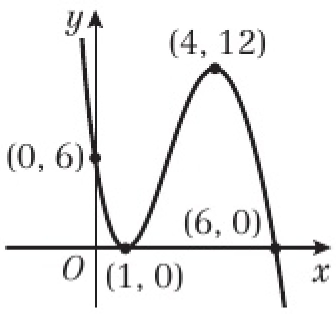
c



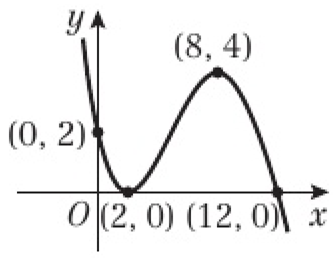
d



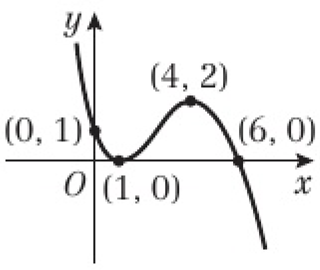
e



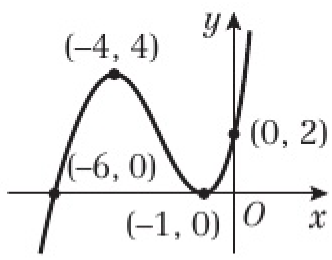
f



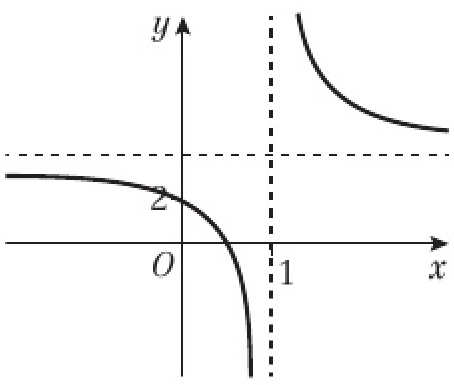
g



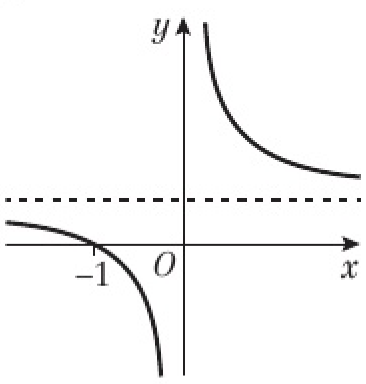
h



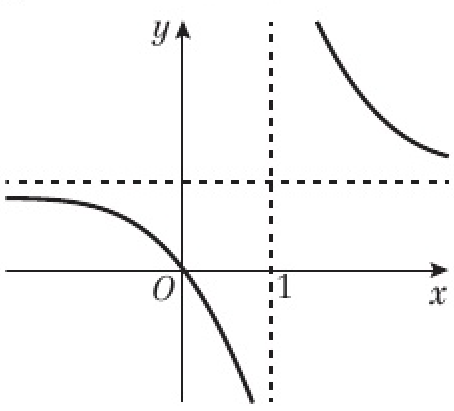
2 **a** $y = 4$, $x = 1$, $(0, 2)$



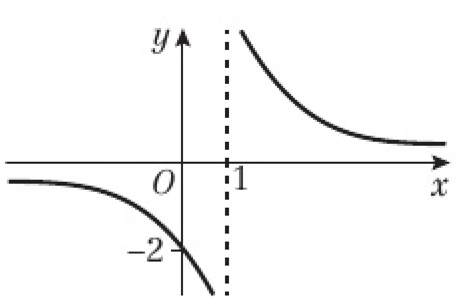
b $y = 2$, $x = 0$, $(-1, 0)$



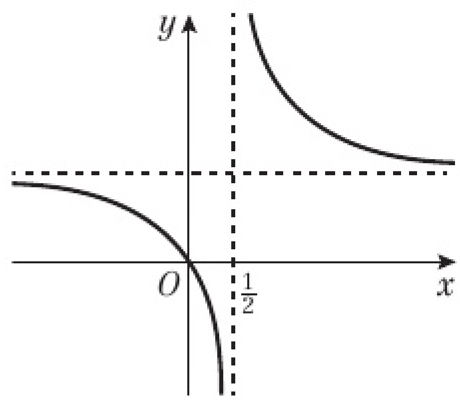
c $y = 4$, $x = 1$, $(0, 0)$



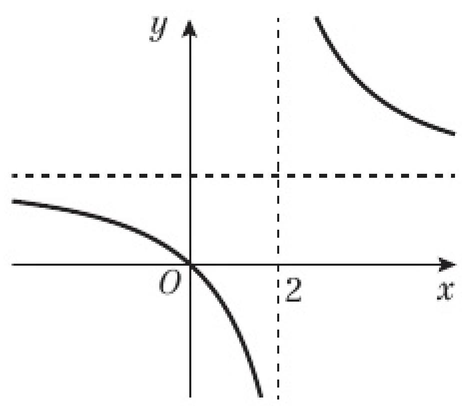
d $y = 0$, $x = 1$, $(0, -2)$



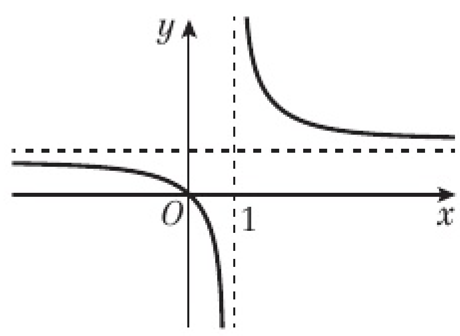
e $y = 2$, $x = \frac{1}{2}$, $(0, 0)$



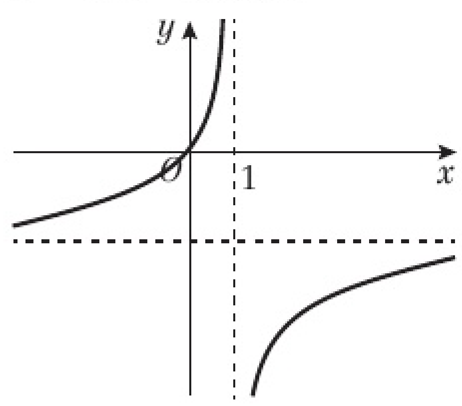
f $y = 2$, $x = 2$, $(0, 0)$



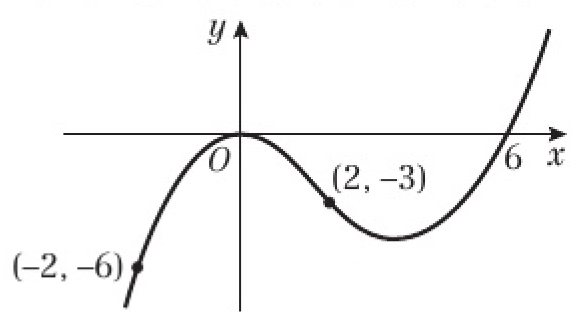
g $y = 1$, $x = 1$, $(0, 0)$



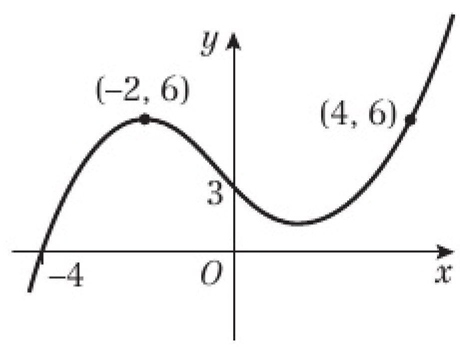
h $y = -2$, $x = 1$, $(0, 0)$



3 **a** $A(-2, -6)$, $B(0, 0)$, $C(2, -3)$, $D(6, 0)$

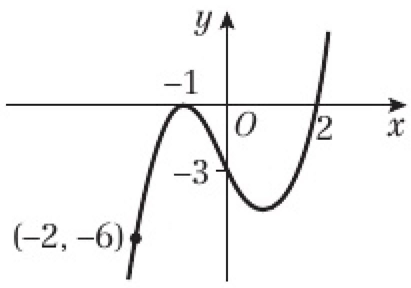


b $A(-4, 0)$, $B(-2, 6)$, $C(0, 3)$, $D(4, 6)$

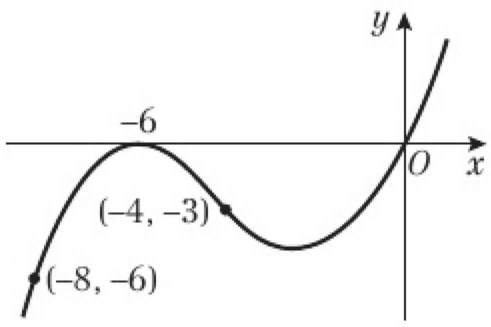


Online Worked solutions are available in SolutionBank.

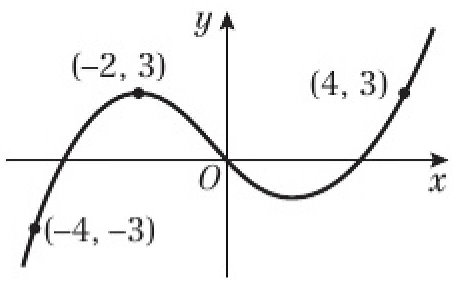
c $A(-2, -6)$, $B(-1, 0)$, $C(0, -3)$, $D(2, 0)$



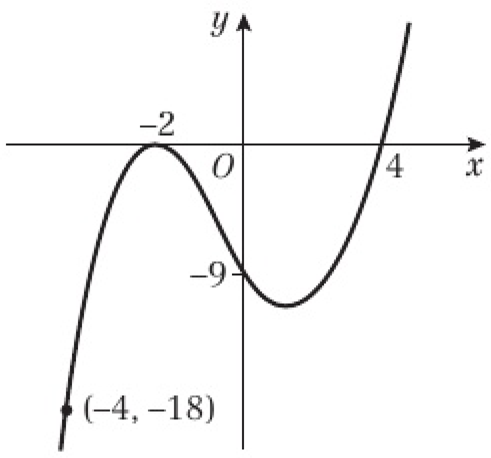
d $A(-8, -6)$, $B(-6, 0)$, $C(-4, -3)$, $D(0, 0)$



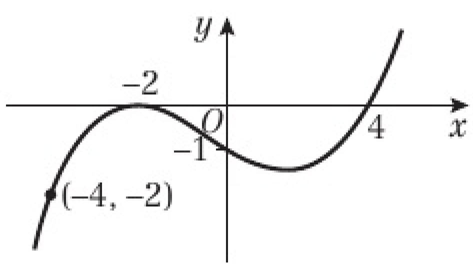
e $A(-4, -3)$, $B(-2, 3)$, $C(0, 0)$, $D(4, 3)$



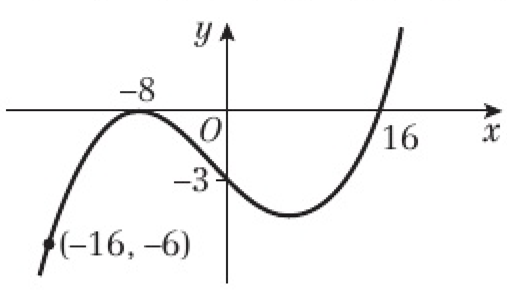
f $A(-4, -18)$, $B(-2, 0)$, $C(0, -9)$, $D(4, 0)$



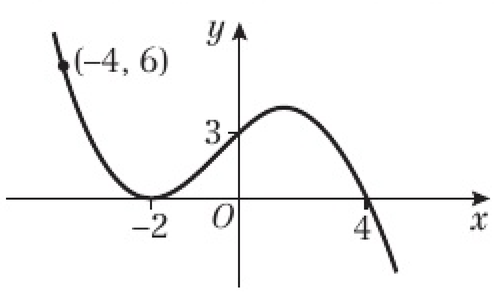
g $A(-4, -2)$, $B(-2, 0)$, $C(0, -1)$, $D(4, 0)$



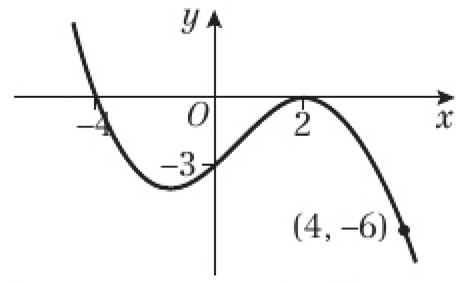
h $A(-16, -6)$, $B(-8, 0)$, $C(0, -3)$, $D(16, 0)$



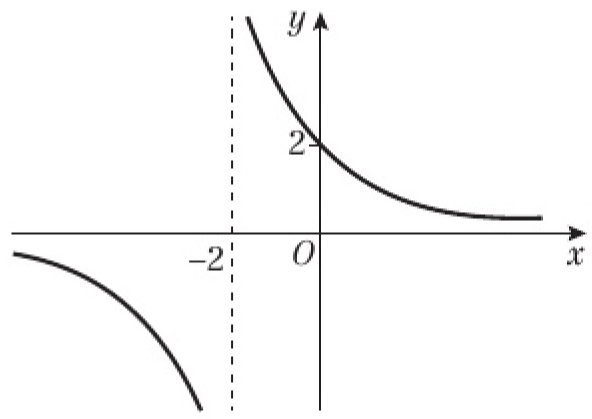
i $A(-4, 6)$, $B(-2, 0)$, $C(0, 3)$, $D(4, 0)$



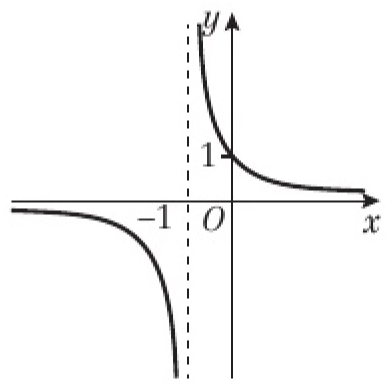
j $A(4, -6)$, $B(2, 0)$, $C(0, -3)$, $D(-4, 0)$



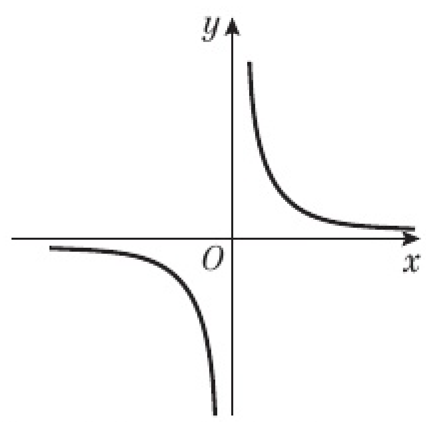
4 a i $x = -2$, $y = 0$, $(0, 2)$



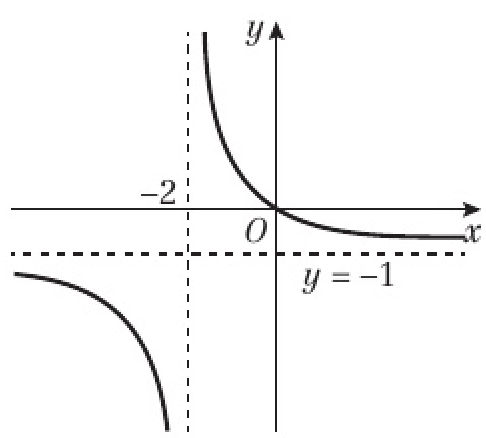
ii $x = -1$, $y = 0$, $(0, 1)$



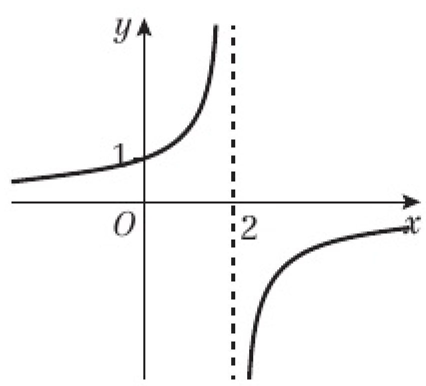
iii $x = 0$, $y = 0$



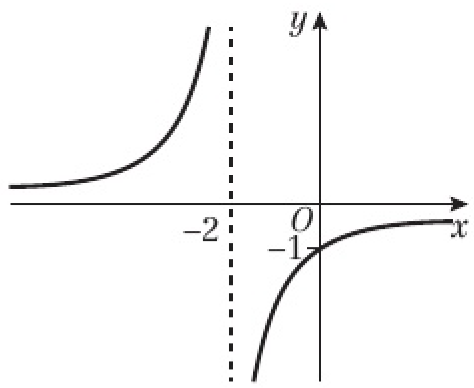
iv $x = -2$, $y = -1$, $(0, 0)$



v $x = 2$, $y = 0$, $(0, 1)$



vi $x = -2$, $y = 0$, $(0, -1)$

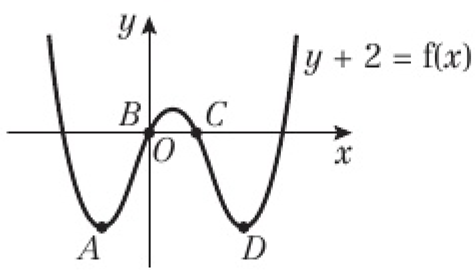


b $\mathrm{f}(x) = \dfrac{2}{x + 2}$

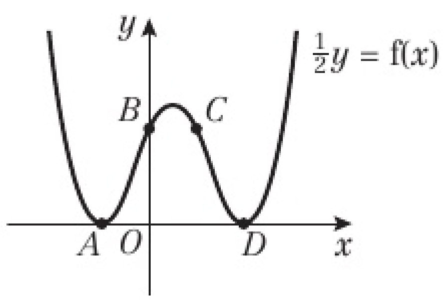
5 **a** $\frac{1}{2}$

b **i** (6, 1) **ii** (2, 3) **iii** (2, −3.5)

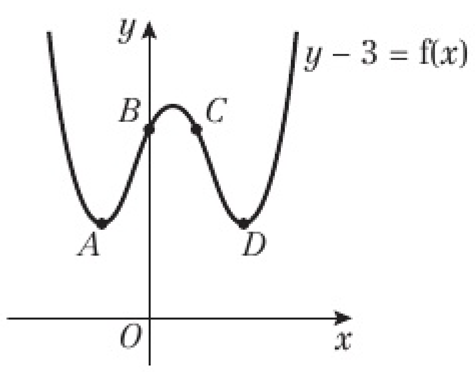
6 **a** $A(-1, -2)$ $B(0, 0)$ $C(1, 0)$ $D(2, -2)$



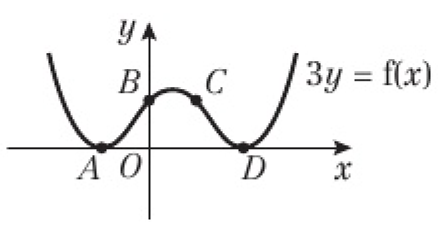
b $A(-1, 0)$ $B(0, 4)$ $C(1, 4)$ $D(2, 0)$



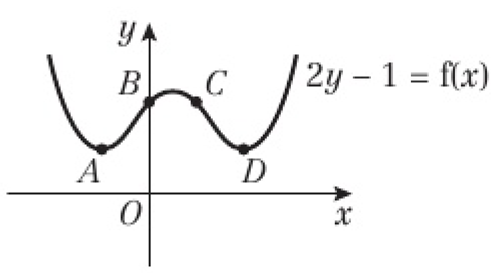
c $A(-1, 3)$ $B(0, 5)$ $C(1, 5)$ $D(2, 3)$



d $A(-1, 0)$ $B(0, \frac{2}{3})$ $C(1, \frac{2}{3})$ $D(2, 0)$

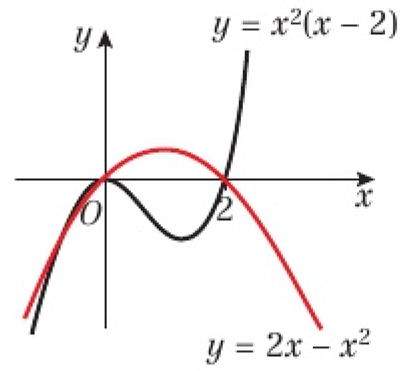


e $A(-1, 0.5)$ $B(0, 1.5)$ $C(1, 1.5)$ $D(2, 0.5)$



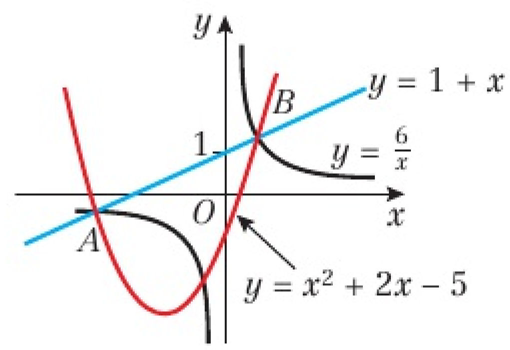
Chapter review 4

1 **a**



b $x = 0, -1, 2$; points $(0, 0)$, $(2, 0)$, $(-1, -3)$

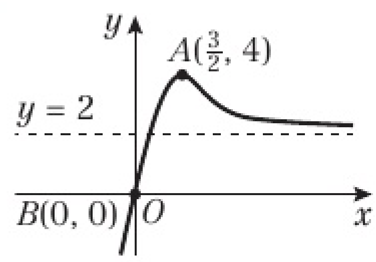
2 **a**



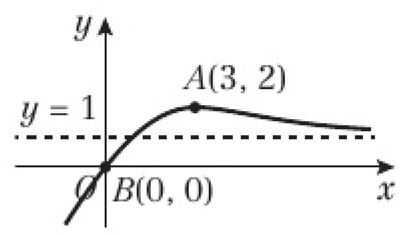
b $A(-3, -2)$, $B(2, 3)$

c $y = x^2 + 2x - 5$

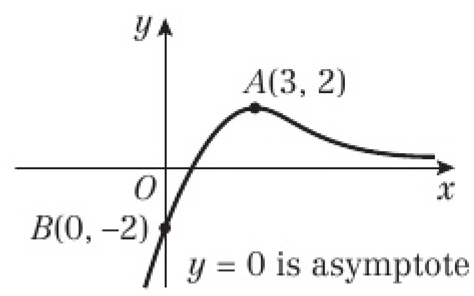
3 **a**



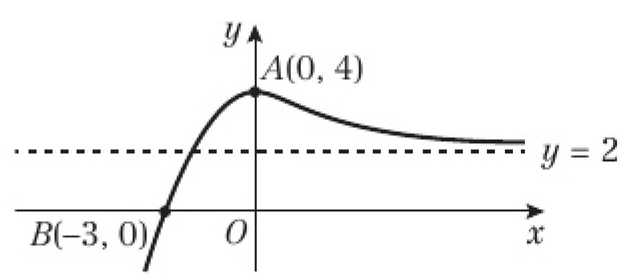
b



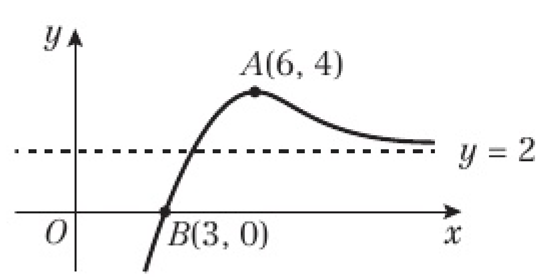
c



d

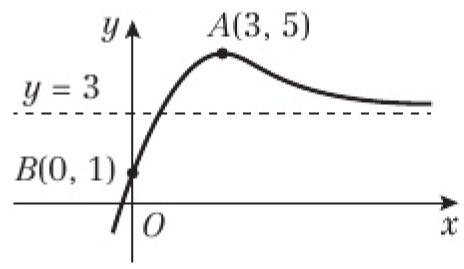


e



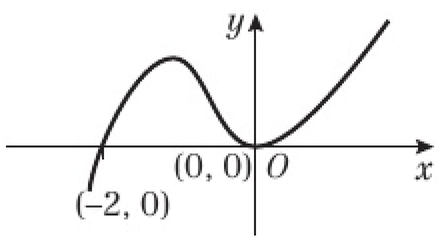
Online Worked solutions are available in SolutionBank.

f

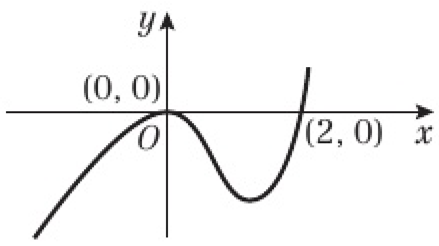


4 **a** $x = -1$ at A, $x = 3$ at B

5 **a**

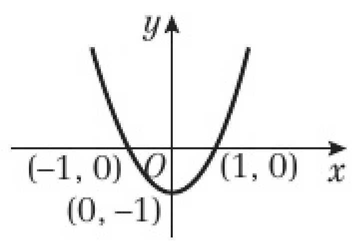


b

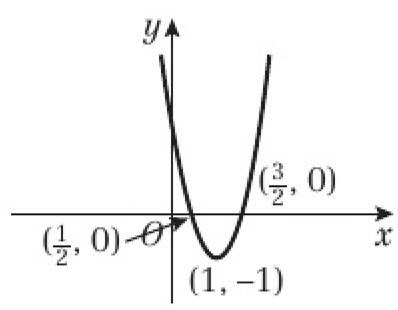


6 **a** $y = x^2 - 4x + 3$

b **i**



ii



7 **a** (0, 2) **b** –2 **c** –1, 1, 2

8 **a** **i** $(\frac{4}{3}, 3)$ **ii** (4, 6) **iii** (9, 3)

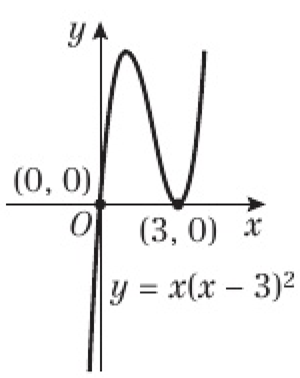
iv (4, –3) **v** $(4, -\frac{1}{2})$

b $f(2x)$, $f(x + 2)$

c **i** $f(x - 4) + 3$ **ii** $2f(\frac{1}{2}x)$

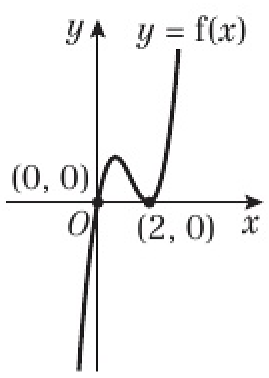
9 **a** $x(x - 3)^2$

b

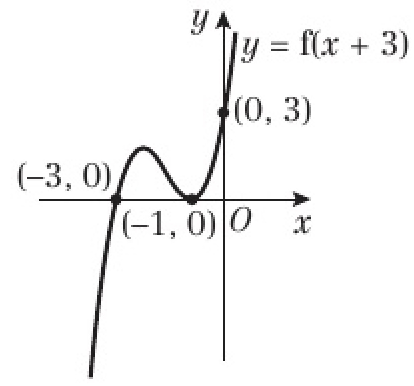


c –4 and –7

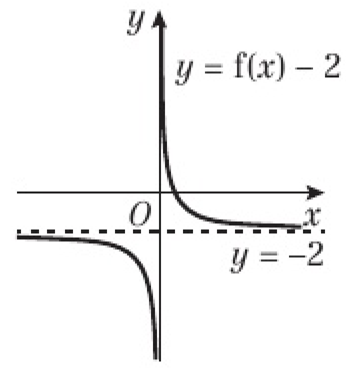
10 **a**



b

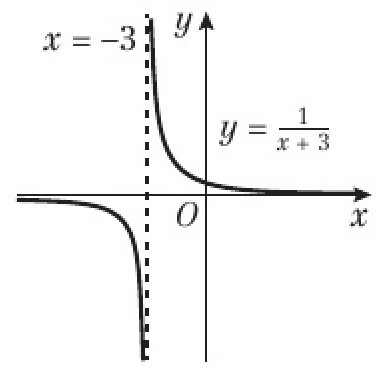


11 **a** Asymptotes at $x = 0$ and $y = -2$



b $(\frac{1}{2}, 0)$

c



d Asymptotes at $y = 0$ and $x = -3$; intersection at $(0, \frac{1}{3})$

Challenge

$(6 - c, -4 - d)$

Review exercise 1

1 **a** 2 **b** $\frac{1}{4}$

2 **a** 625 **b** $\frac{4}{3}x^{\frac{2}{3}}$

3 **a** $4\sqrt{5}$ **b** $21 - 8\sqrt{5}$

4 **a** 13 **b** $8 - 2\sqrt{3}$

5 **a** $1 + 2\sqrt{k}$ **b** $1 + 6\sqrt{k}$

6 **a** $25x^{-4}$ **b** x^2

7 $8 + 8\sqrt{2}$

8 $1 - 2\sqrt{2}$

9 **a** $(x - 8)(x - 2)$ **b** $y = 1, y = \frac{1}{3}$

10 **a** $a = -4, b = -45$ **b** $x = 4 \pm 3\sqrt{5}$

11 4.19 (3 s.f.)

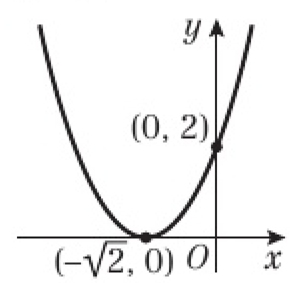
12 **a** $(x - 3)^2 + 9$

b P is (0, 18), Q is (3, 9)

c $x = 3 + 4\sqrt{2}$

13 **a** $k = 2$

b



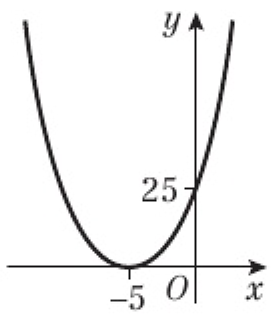
14 **a** $x^3(x^3 - 8)(x^3 + 1)$ **b** –1, 0, 2

15 **a** $a = 5, b = 11$

b $(x + 5)^2 = -11$, so no real roots

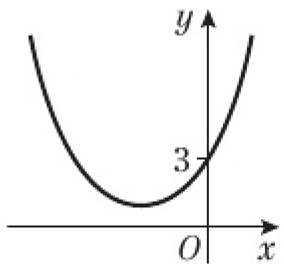
c $k = 25$

d



16 a $a = 1, b = 2$

b



c discriminant = –8, so no real roots

d $-2\sqrt{3} < k < 2\sqrt{3}$

17 a Substitute $y = x - 4$ into $2x^2 - xy = 8$ and rearrange.

b $x = -2 \pm 2\sqrt{3}, y = -6 \pm 2\sqrt{3}$

18 a $x > \frac{1}{4}$

b $x < \frac{1}{2}$ or $x > 3$

c $\frac{1}{4} < x < \frac{1}{2}$ or $x > 3$

19 $-2(x + 1) = x^2 - 5x + 2$

$x^2 - 3x + 4 = 0$

The discriminant of this is $-7 < 0$, so no real solutions.

20 a $x = \frac{7}{2}, y = -2; x = -3, y = 11$

b $x < -3$ or $x > 3\frac{1}{2}$

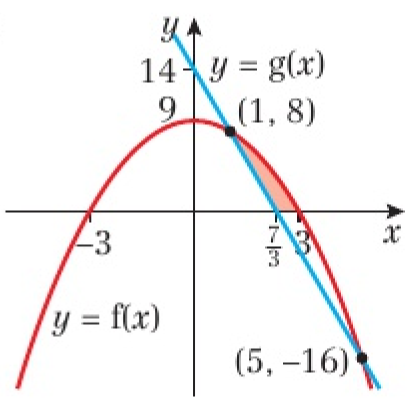
21 a Different real roots, discriminant > 0

so $k^2 - 4k - 12 > 0$

b $k < -2$ or $k > 6$

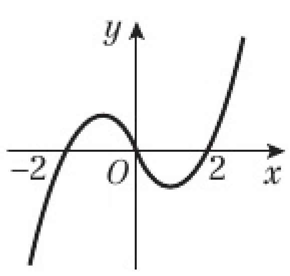
22 $x < -5$ or $x > -2$

23 a, b

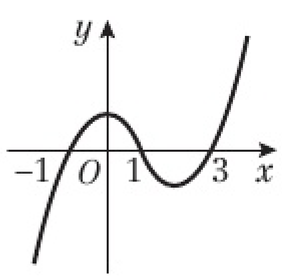


24 a $x(x - 2)(x + 2)$

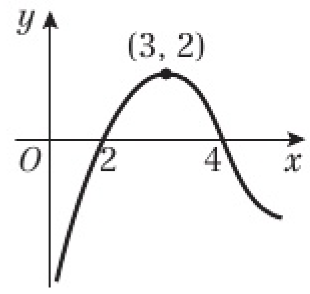
b



c

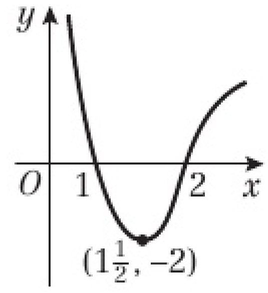


25 a



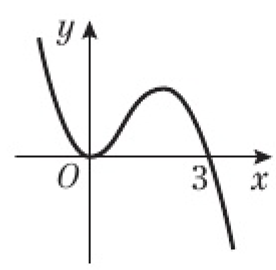
(2, 0) (4, 0) and (3, 2)

b



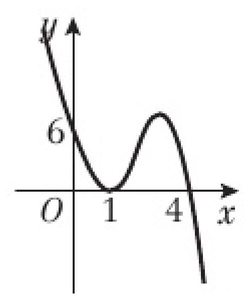
(1, 0) (2, 0) and $(1\frac{1}{2}, -2)$

26 a



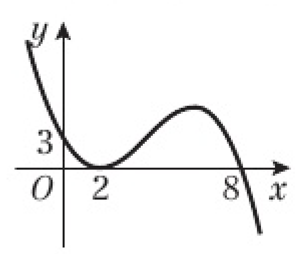
(0, 0) and (3, 0)

b



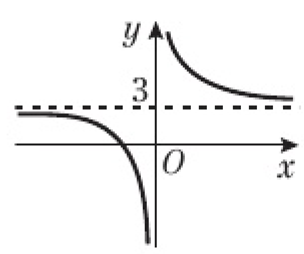
(1, 0) (4, 0) and (0, 6)

c



(2, 0) (8, 0) and (0, 3)

27 a

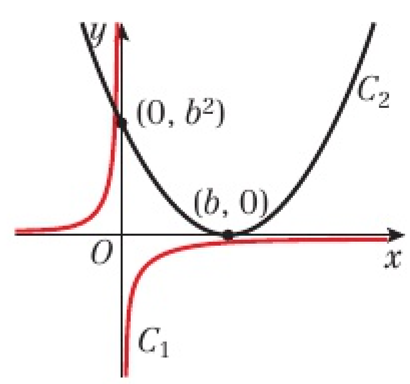


Asymptotes: $y = 3$ and $x = 0$

b $(-\frac{1}{3}, 0)$

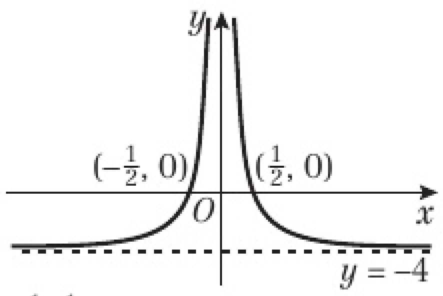
28 a (6, 8) **b** (9, –8) **c** (6, –4)

29 a



b 1

30 **a**



b $-\frac{1}{2}, \frac{1}{2}$

Challenge

1 **a** $x = 1, x = 9$ **b** $x = 0, x = 2$

2 $\sqrt{2}$ cm, $3\sqrt{2}$ cm

3 $3x^3 + x^2 - x = 2x(x - 1)(x + 1)$
$3x^3 + x^2 - x = 2x^3 - 2x$
$x^3 + x^2 + x = 0$
$x(x^2 + x + 1) = 0$
The discriminant of the bracket is $-3 < 0$ so this contributes no real solutions.
The only solution is when $x = 0$ at $(0, 0)$.

CHAPTER 5

Prior knowledge check

1 **a** $(-2, -1)$ **b** $\left(\frac{9}{19}, \frac{26}{19}\right)$ **c** $(7, 3)$

2 **a** $4\sqrt{5}$ **b** $10\sqrt{2}$ **c** $5\sqrt{5}$

3 **a** $y = 5 - 2x$ **b** $y = \frac{2}{5}x - \frac{9}{5}$ **c** $y = \frac{3}{7}x + \frac{12}{7}$

Exercise 5A

1 **a** $\frac{1}{2}$ **b** $\frac{1}{6}$ **c** $-\frac{3}{5}$ **d** 2
e −1 **f** $\frac{1}{2}$ **g** $\frac{1}{2}$ **h** 8
i $\frac{2}{3}$ **j** −4 **k** $-\frac{1}{3}$ **l** $-\frac{1}{2}$
m 1 **n** $\frac{q^2 - p^2}{q - p} = q + p$

2 7
3 12
4 $4\frac{1}{3}$
5 $2\frac{1}{4}$
6 $\frac{1}{4}$
7 26
8 −5
9 Gradient of AB = gradient of BC = 0.5; point B is common
10 Gradient of AB = gradient of BC = −0.5; point B is common

Exercise 5B

1 **a** −2 **b** −1 **c** 3 **d** $\frac{1}{3}$
e $-\frac{2}{3}$ **f** $\frac{5}{4}$ **g** $\frac{1}{2}$ **h** 2
i $\frac{1}{2}$ **j** $\frac{1}{2}$ **k** −2 **l** $-\frac{3}{2}$

2 **a** 4 **b** −5 **c** $-\frac{2}{3}$ **d** 0
e $\frac{7}{5}$ **f** 2 **g** 2 **h** −2
i 9 **j** −3 **k** $\frac{3}{2}$ **l** $-\frac{1}{2}$

3 **a** $4x - y + 3 = 0$ **b** $3x - y - 2 = 0$
c $6x + y - 7 = 0$ **d** $4x - 5y - 30 = 0$
e $5x - 3y + 6 = 0$ **f** $7x - 3y = 0$
g $14x - 7y - 4 = 0$ **h** $27x + 9y - 2 = 0$
i $18x + 3y + 2 = 0$ **j** $2x + 6y - 3 = 0$
k $4x - 6y + 5 = 0$ **l** $6x - 10y + 5 = 0$

4 (3, 0)
5 (0, 0)
6 (0, 5), (−4, 0)
7 **a** $\frac{1}{3}$ **b** $x - 3y + 15 = 0$
8 **a** $-\frac{2}{5}$ **b** $2x + 5y - 10 = 0$
9 $ax + by + c = 0$
$by = -ax - c$
$y = \left(-\frac{a}{b}\right)x - \left(\frac{c}{b}\right)$
10 $a = 6, c = 10$
11 $P(3,0)$
12 **a** −16 **b** −27

Challenge

Gradient $= -\frac{a}{b}$; y-intercept $= a$. So $y = -\frac{a}{b}x + a$

Rearrange to give $ax + by - ab = 0$

Exercise 5C

1 **a** $y = 2x + 1$ **b** $y = 3x + 7$ **c** $y = -x - 3$
d $y = -4x - 11$ **e** $y = \frac{1}{2}x + 12$ **f** $y = -\frac{2}{3}x - 5$
g $y = 2x$ **h** $y = -\frac{1}{2}x + 2b$

2 **a** $y = 4x - 4$ **b** $y = x + 2$ **c** $y = 2x + 4$
d $y = 4x - 23$ **e** $y = x - 4$ **f** $y = \frac{1}{2}x + 1$
g $y = -4x - 9$ **h** $y = -8x - 33$ **i** $y = \frac{6}{5}x$
j $y = \frac{2}{7}x + \frac{5}{14}$

3 $5x + y - 37 = 0$
4 $y = x + 2$, $y = -\frac{1}{6}x - \frac{1}{3}$, $y = -6x + 23$
5 $a = 3, c = -27$
6 $a = -4, b = 8$

Challenge

a $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$

b $y - y_1 = \frac{(y_2 - y_1)}{(x^2 - y_1)}(x - x_1)$

$\frac{(y - y_1)}{(y_2 - y_1)} = \frac{(x - x_1)}{(x_2 - x_1)}$

c $y = \frac{3}{7}x + \frac{52}{7}$

Exercise 5D

1 $y = 3x - 6$ **2** $y = 2x + 8$
3 $2x - 3y + 24 = 0$ **4** $-\frac{1}{5}$
5 (−3, 0) **6** (0, 1)
7 $(0, 3\frac{1}{2})$ **8** $y = \frac{2}{5}x + 3$
9 $2x + 3y - 12 = 0$ **10** $\frac{8}{5}$
11 $y = \frac{4}{3}x - 4$ **12** $6x + 15y - 10 = 0$
13 $y = -\frac{4}{5}x + 4$ **14** $x - y + 5 = 0$
15 $y = -\frac{3}{8}x + \frac{1}{2}$ **16** $y = 4x + 13$

Exercise 5E

1 **a** Parallel **b** Not parallel **c** Not parallel

2 r: $y = \frac{4}{5}x + 3.2$, s: $y = \frac{4}{5}x - 7$
Gradients equal therefore lines are parallel.

3 Gradient of $AB = \frac{3}{5}$, gradient of $BC = -\frac{7}{2}$, gradient of $CD = \frac{3}{5}$, gradient of $AD = \frac{10}{3}$. The quadrilateral has a pair of parallel sides, so it is a trapezium.

4 $y = 5x + 3$

5 $2x + 5y + 20 = 0$
6 $y = -\frac{1}{2}x + 7$
7 $y = \frac{2}{3}x$
8 $4x - y + 15 = 0$

Exercise 5F

1 a Perpendicular b Parallel
c Neither d Perpendicular
e Perpendicular f Parallel
g Parallel h Perpendicular
i Perpendicular j Parallel
k Neither l Perpendicular

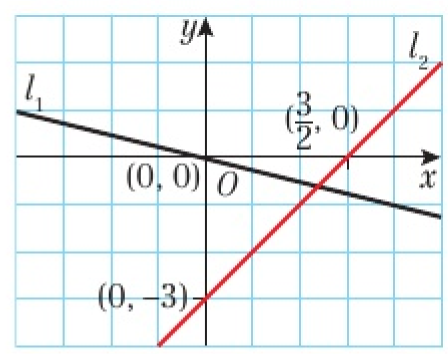
2 $y = -\frac{1}{6}x + 1$
3 $y = \frac{8}{3}x - 8$
4 $y = -\frac{1}{3}x$
5 $y = -\frac{1}{3}x + \frac{13}{3}$
6 $y = -\frac{3}{2}x + \frac{17}{2}$
7 $3x + 2y - 5 = 0$
8 $7x - 4y + 2 = 0$
9 l has gradient $-\frac{1}{3}$ and n has gradient 3. Gradients are negative reciprocals, therefore lines perpendicular.
10 AB: $y = -\frac{1}{2}x + 4\frac{1}{2}$, CD: $y = -\frac{1}{2}x - \frac{1}{2}$, AD: $y = 2x + 7$, BC: $y = 2x - 13$. Two pairs of parallel sides and lines with gradients 2 and $-\frac{1}{2}$ are perpendicular, so $ABCD$ is a rectangle.
11 a $A(\frac{7}{5}, 0)$ b $55x - 25y - 77 = 0$
12 $-\frac{9}{4}$

Exercise 5G

1 a 10 b 13 c 5 d $\sqrt{5}$
e $\sqrt{106}$ f $\sqrt{113}$
2 Distance between A and $B = \sqrt{50}$ and distance between B and $C = \sqrt{50}$ so the lines are congruent.
3 Distance between P and $Q = \sqrt{74}$ and distance between Q and $R = \sqrt{73}$ so the lines are not congruent.
4 $x = -8$ or $x = 6$
5 $y = -2$ or $y = 16$
6 a Both lines have gradient 2.
b $y = -\frac{1}{2}x + \frac{23}{2}$ or $x + 2y - 23 = 0$
c $\left(\frac{29}{5}, \frac{43}{5}\right)$
d $\frac{7\sqrt{5}}{5}$
7 $P\left(-\frac{3}{5}, \frac{29}{5}\right)$ or $P(3, -5)$
8 a $AB = \sqrt{178}$, $BC = 3$ and $AC = \sqrt{205}$. All sides are different lengths, therefore the triangle is a scalene triangle.
b $\frac{39}{2}$ or 19.5
9 a $A(2, 11)$ b $B\left(\frac{41}{4}, 0\right)$ c $\frac{451}{8}$
10 a $\left(\frac{5}{2}, 0\right)$ b $(-5, 0)$
c $(-10, -10)$ d $\frac{75}{2}$
11 a $y = \frac{1}{2}x - \frac{9}{2}$ b $y = -2x + 8$
c $T(0, 8)$ d $RS = 2\sqrt{5}$ and $TR = 5\sqrt{5}$
e 25
12 a $x + 4y - 52 = 0$ b $A(0,13)$
c $B(4, 12)$ d 26

Chapter review 5

1 a $y = -\frac{5}{12}x + \frac{11}{6}$ b -22
2 a $\frac{2k-2}{8-k} = \frac{1}{3}$ therefore $7k = 14$, $k = 2$
b $y = \frac{1}{3}x + \frac{1}{3}$
3 a $L_1 = y = \frac{1}{7}x + \frac{12}{7}$, $L_2 = y = -x + 12$
b $(9, 3)$
4 a $y = \frac{3}{2}x - \frac{3}{2}$ b $(3, 3)$
5 $11x - 10y + 19 = 0$
6 a $y = -\frac{1}{2}x + 3$ b $y = \frac{1}{4}x + \frac{9}{4}$
7 Gradient $= \frac{3 + 4\sqrt{3} - 3\sqrt{3}}{2 + \sqrt{3} - 1} = \frac{3 + \sqrt{3}}{1 + \sqrt{3}} = \sqrt{3}$
$y = \sqrt{3}x + c$ and $A(1, 3\sqrt{3})$, so $c = 2\sqrt{3}$
Equation of line is $y = \sqrt{3}x + 2\sqrt{3}$
When $y = 0$, $x = -2$, so the line meets the x-axis at $(-2, 0)$
8 a $y = -3x + 14$ b $(0, 14)$
9 a $y = -\frac{1}{2}x + 4$ b Students own work.
c $(1, 1)$. Note: equation of line n: $y = -\frac{1}{2}x + \frac{3}{2}$
10 20
11 a $2x + y = 20$ b $y = \frac{1}{3}x + \frac{4}{3}$
12 a $\frac{1}{2}$ b 6 c $2x + y - 16 = 0$
d 10
13 a $7x + 5y - 18 = 0$ b $\frac{162}{35}$
14 a



b $\left(\frac{4}{3}, -\frac{1}{3}\right)$ c $12x - 3y - 17 = 0$
15 a $x + 2y - 16 = 0$
b $y = -\frac{2}{3}x$
c $C(-48, 32)$
d Slope of OA is $\frac{3}{2}$. Slope of OC is $-\frac{2}{3}$. Lines are perpendicular.
e $OA = 2\sqrt{13}$ and $OC = 16\sqrt{13}$
f Area = 208
16 a $d = \sqrt{50a^2} = 5a\sqrt{2}$ b $5\sqrt{2}$
c $15\sqrt{2}$ d $25\sqrt{2}$
17 a $d = \sqrt{10x^2 - 28x + 26}$
b $B\left(-\frac{6}{5}, -\frac{18}{5}\right)$ and $C(4, 12)$
c $y = -\frac{1}{3}x + \frac{14}{3}$
d $\left(\frac{7}{5}, \frac{21}{5}\right)$
e 20.8

Challenge

1 130
2 $\left(\frac{78}{19}, \frac{140}{19}\right)$
3 $\left(a, \frac{a(c-a)}{b}\right)$

CHAPTER 6

Prior knowledge check

1 **a** 3.10 cm **b** 9.05 cm
2 **a** 25.8° **b** 77.2°
3 **a** graph of $x^2 + 3x$ **b** graph of $(x + 2)^2 + 3(x + 2)$
c graph of $x^2 + 3x - 3$ **d** graph of $(0.5x)^2 + 3(0.5x)$

Exercise 6A

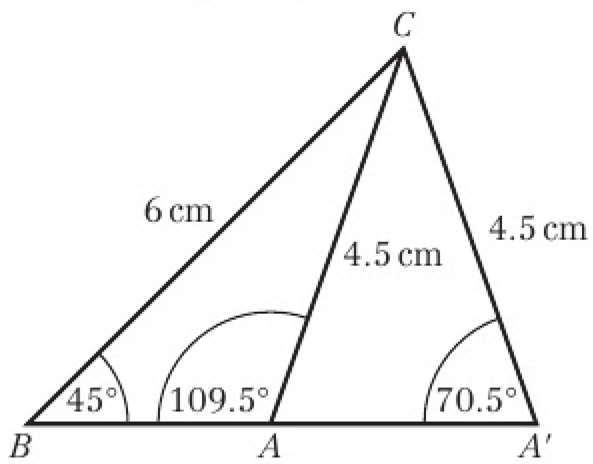
1 **a** 3.19 cm **b** 1.73 cm ($\sqrt{3}$ cm) **c** 9.85 cm
d 4.31 cm **e** 6.84 cm **f** 9.80 cm
2 **a** 108(.2)° **b** 90° **c** 60°
d 52.6° **e** 137° **f** 72.2°
3 192 km
4 11.2 km
5 128.5° or 031.5° (Angle BAC = 48.5°)
6 302 yards (301.5…)
7 Using the cosine rule $\frac{5^2 + 4^2 - 6^2}{2 \times 5 \times 4} = \frac{1}{8}$
8 Using the cosine rule $\frac{2^2 + 3^2 - 4^2}{2 \times 2 \times 3} = -\frac{1}{4}$
9 ACB = 22.3°
10 ABC = 108(.4)°
11 104(.48)°
12 x = 4.4 cm
13 x = 42 cm
14 **a** $y^2 = (5 - x)^2 + (4 + x)^2 - 2(5 - x)(4 + x)\cos 120°$
$= 25 - 10x + x^2 + 16 + 8x + x^2 - 2(20 + x - x^2)\left(-\frac{1}{2}\right)$
$= x^2 - x + 61$
b Minimum AC^2 = 60.75; it occurs for $x = \frac{1}{2}$
15 **a** $\cos\angle ABC = \frac{x^2 + 5^2 - (10 - x)^2}{2x \times 5}$
$= \frac{20x - 75}{10x} = \frac{4x - 15}{2x}$
b 3.5
16 65.3°
17 **a** 28.7 km **b** 056.6°

Exercise 6B

1 **a** 15.2 cm **b** 9.57 cm **c** 8.97 cm **d** 4.61 cm
2 **a** x = 84°, y = 6.32
b x = 13.5, y = 16.6
c x = 85°, y = 13.9
d x = 80°, y = 6.22 (isosceles triangle)
e x = 6.27, y = 7.16
f x = 4.49, y = 7.49 (right-angled)
3 **a** 36.4° **b** 35.8° **c** 40.5° **d** 130°
4 **a** 48.1° **b** 45.6° **c** 14.8° **d** 48.7°
e 86.5° **f** 77.4°
5 **a** 1.41 cm ($\sqrt{2}$ cm) **b** 1.93 cm
6 QPR = 50.6°, PQR = 54.4°
7 **a** x = 43.2°, y = 5.02 cm **b** x = 101°, y = 15.0 cm
c x = 6.58 cm, y = 32.1° **d** x = 54.6°, y = 10.3 cm
e x = 21.8°, y = 3.01 **f** x = 45.9°, y = 3.87°
8 **a** 6.52 km **b** 3.80 km
9 **a** 7.31 cm **b** 1.97 cm
10 **a** 66.3° **b** 148 m
11 Using the sine rule, $x = \frac{4\sqrt{2}}{2 + \sqrt{2}}$; rationalising
$x = \frac{4\sqrt{2}(2 - \sqrt{2})}{2} = 4\sqrt{2} - 4 = 4(\sqrt{2} - 1)$.
12 **a** 36.5 m
b That the angles have been measured from ground level

Exercise 6C

1 **a** 70.5°, 109° (109.5°)
b



2 **a** x = 74.6°, y = 65.4°
x = 105°, y = 34.6°
b x = 59.8°, y = 48.4 cm
x = 120°, y = 27.3cm
c x = 56.8°, y = 4.37 cm
x = 23.2°, y = 2.06 cm
3 **a** 5 cm (ACB = 90°) **b** 24.6°
c 45.6°, 134(.4)°
4 2.97 cm
5 In one triangle ABC = 101° (100.9°); in the other BAC = 131° (130.9°)
6 **a** 62.0° **b** The swing is symmetrical

Exercise 6D

1 **a** 23.7 cm² **b** 4.31 cm² **c** 20.2 cm²
2 **a** x = 41.8° or 138(.2)°
b x = 26.7° or 153(.3)°
c x = 60° or 120°
3 275(.3) m (third side = 135.3 m)
4 3.58
5 **a** Area $= \frac{1}{2}(x + 2)(5 - x)\sin 30°$
$= \frac{1}{2}(10 + 3x - x^2) \times \frac{1}{2}$
$= \frac{1}{4}(10 + 3x - x^2)$
b Maximum $A = 3\frac{1}{16}$, when $x = 1\frac{1}{2}$
6 **a** $\frac{1}{2}x(5 + x)\sin 150° = \frac{15}{4}$
$\frac{1}{2}(5x + x^2) \times \frac{1}{2} = \frac{15}{4}$
$5x + x^2 = 15$
$x^2 + 5x - 15 = 0$
b 2.11

Exercise 6E

1 **a** x = 37.7°, y = 86.3°, z = 6.86
b x = 48°, y = 19.5, z = 14.6
c x = 30°, y = 11.5, z = 11.5
d x = 21.0°, y = 29.0°, z = 8.09
e x = 93.8°, y = 56.3°, z = 29.9°
f x = 97.2°, y = 41.4°, z = 41.4°
g x = 45.3°, y = 94.7°, z = 14.7
or x = 135°, y = 5.27°, z = 1.36
h x = 7.07, y = 73.7°, z = 61.3°
or x = 7.07, y = 106°, z = 28.7°
i x = 49.8°, y = 9.39, z = 37.0°
2 **a** ACB = 32.4°, ABC = 108°, AC = 15.1 cm
Area = 41.3 cm²
b BAC = 41.5°, ABC = 28.5°, AB = 9.65 cm
Area = 15.7 cm²

3 **a** 8 km
b 060°

4 107 km

5 12 km

6 **a** 5.44 **b** 7.95 **c** 36.8°

7 **a** $AB + BC > AC \Rightarrow x + 6 > 7 \Rightarrow x > 1$;
$AC + AB > BC \Rightarrow 11 > x + 2 \Rightarrow x < 9$
b **i** $x = 6.08$ from $x^2 = 37$
Area = 14.0 cm²
ii $x = 7.23$ from $x^2 - 4(\sqrt{2} - 1)x - (29 + 8\sqrt{2}) = 0$
Area = 13.1 cm²

8 **a** $x = 4$ **b** 4.68 cm²

9 $AC = 1.93$ cm

10 **a** $AC^2 = (2 - x)^2 + (x + 1)^2 - 2(2 - x)(x + 1)\cos 120°$
$= (4 - 4x + x^2) + (x^2 + 2x + 1) - 2(-x^2 + x + 2)\left(-\frac{1}{2}\right)$
$= x^2 - x + 7$
b $\frac{1}{2}$

11 $4\sqrt{10}$

12 $AC = 1\frac{2}{3}$ cm and $BC = 6\frac{1}{3}$ cm
Area = 5.05 cm²

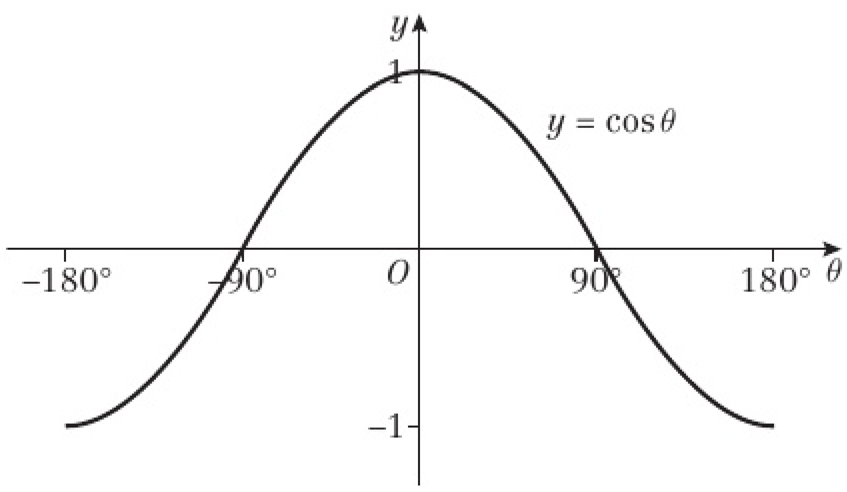
13 **a** 61.3° **b** 78.9 cm²

14 **a** $DAB = 136.3°$, $BCD = 50.1°$
b 13.1 m²
c 5.15 m

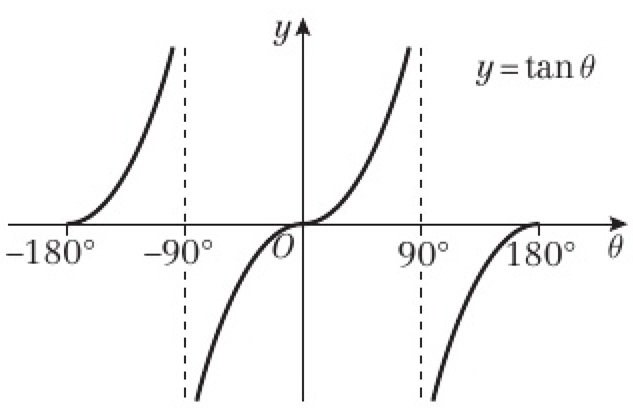
15 34.2 cm²

Exercise 6F

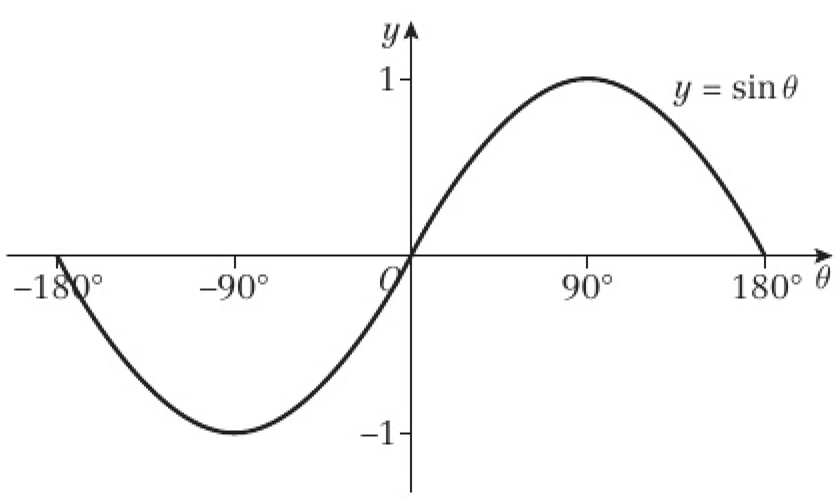
1



2



3

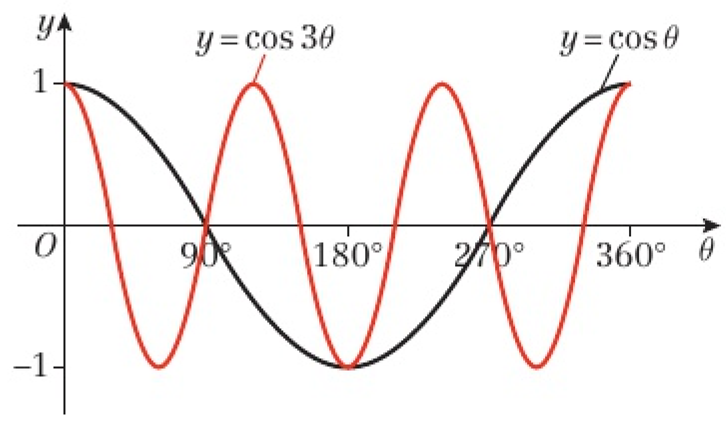


4 **a** −30°
b **i** −120° **ii** −60°, 120°
c **i** 135° **ii** −45°, −135°

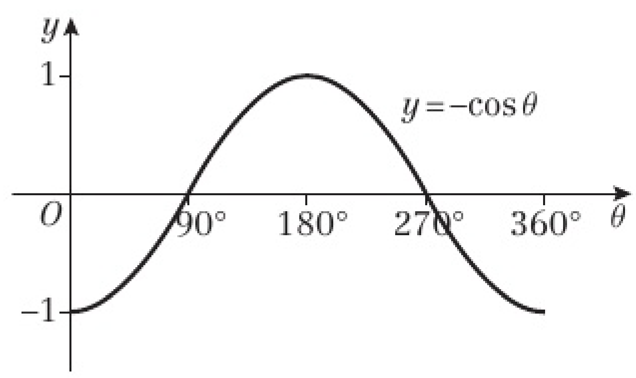
Exercise 6G

1 **a** **i** 1, $x = 0°$ **ii** −1, $x = 180°$
b **i** 4, $x = 90°$ **ii** −4, $x = 270°$
c **i** 1, $x = 0°$ **ii** −1, $x = 180°$
d **i** 4, $x = 90°$ **ii** 2, $x = 270°$
e **i** 1, $x = 270°$ **ii** −1, $x = 90°$
f **i** 1, $x = 30°$ **ii** −1, $x = 90°$

2

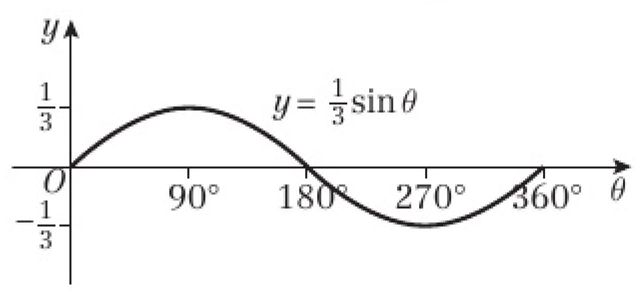


3 **a** The graph of $y = -\cos\theta$ is the graph of $y = \cos\theta$ reflected in the θ-axis



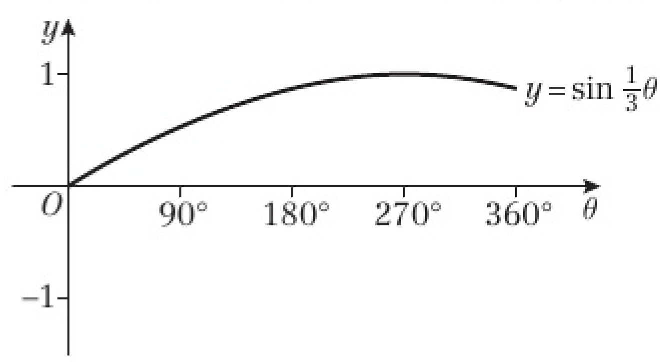
Meets θ-axis at (90°, 0), (270°, 0)
Meets y-axis at (0°, −1)
Maximum at (180°, 1)
Minimum at (0°, −1) and (360°, −1)

b The graph of $y = \frac{1}{3}\sin\theta$ is the graph of $y = \sin\theta$ stretched by a scale factor $\frac{1}{3}$ in the y direction.



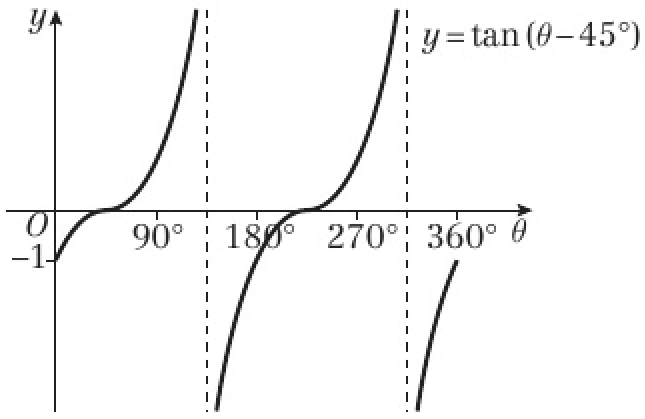
Meets θ-axis at (0°, 0), (180°, 0), (360°, 0)
Meets y-axis at (0°, 0)
Maximum at (90°, $\frac{1}{3}$)
Minimum at (270°, $-\frac{1}{3}$)

c The graph of $y = \sin\frac{1}{3}\theta$ is the graph of $y = \sin\theta$ stretched by a scale factor 3 in the θ direction.



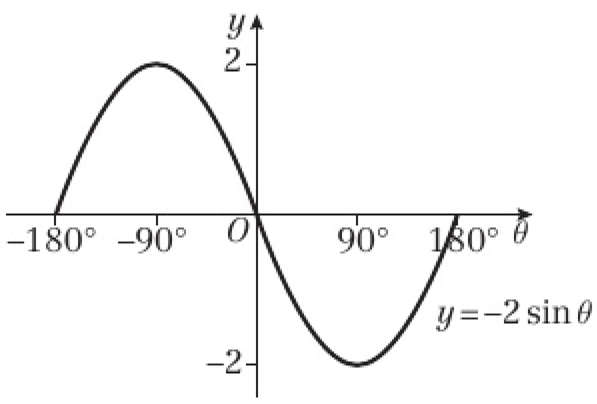
Only meets axis at origin
Maximum at (270°, 1)

d The graph of $y = \tan(\theta - 45°)$ is the graph of $\tan\theta$ translated by 45° in the positive θ direction.



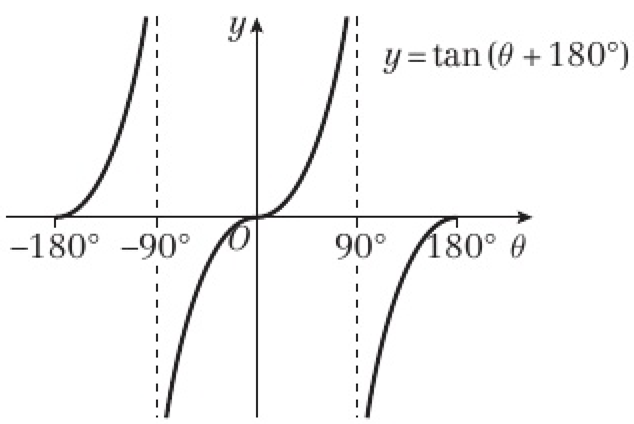
Meets θ-axis at (45°, 0), (225°, 0)
Meets y-axis at (0°, –1)
(Asymptotes at θ = 135° and θ = 315°)

4 **a** This is the graph of $y = \sin\theta$ stretched by scale factor –2 in the y-direction (i.e. reflected in the θ-axis and scaled by 2 in the y-direction).



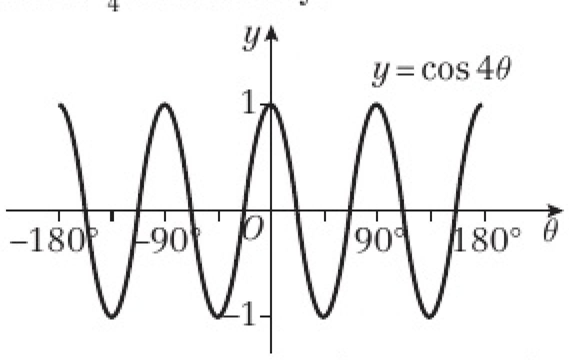
Meets θ-axis at (–180°, 0), (0, 0), (180, 0)
Maximum at (–90°, 2)
Minimum at (90°, –2).

b This is the graph of $y = \tan\theta$ translated by 180° in the negative θ direction.



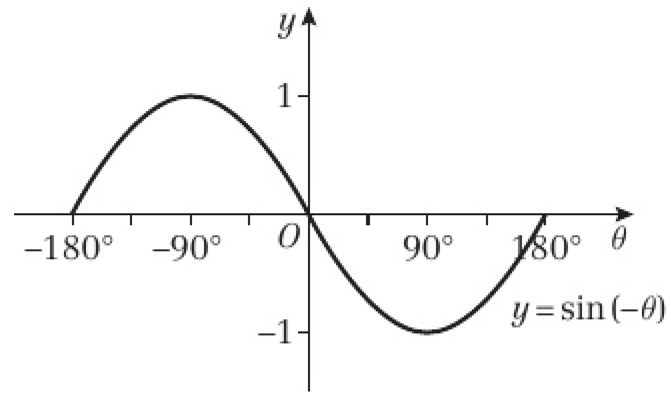
As $\tan\theta$ has a period of 180°
$\tan(\theta + 180°) = \tan\theta$
Meets θ-axis at (–180°, 0), (0, 0), (180°, 0)
Meets y-axis at (0, 0)

c This is the graph of $y = \cos\theta$ stretched by scale factor $\frac{1}{4}$ horizontally.



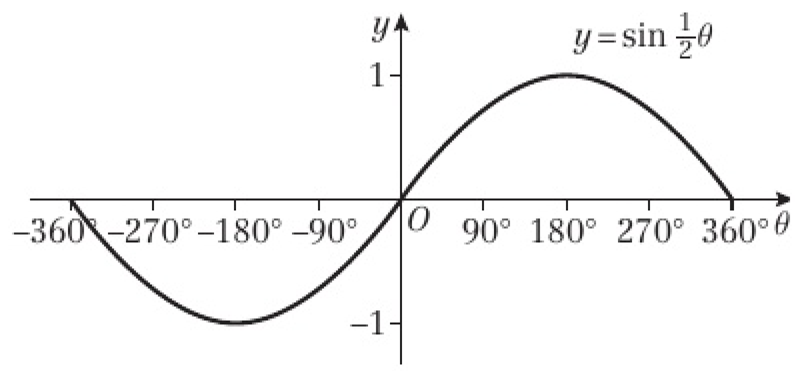
Meets θ-axis at $(-157\frac{1}{2}°, 0)$, $(-112\frac{1}{2}°, 0)$, $(-67\frac{1}{2}°, 0)$, $(-22\frac{1}{2}°, 0)$, $(22\frac{1}{2}°, 0)$, $(67\frac{1}{2}°, 0)$, $(112\frac{1}{2}°, 0)$, $(157\frac{1}{2}°, 0)$
Meets y-axis at (0, 1)
Maxima at (–180°, 1), (–90°, 1), (0, 1), (90°, 1), (180°, 1)
Minima at (–135°, –1), (–45°, –1), (45°, –1), (135°, –1)

d This is the graph of $y = \sin\theta$ reflected in the y-axis. (This is the same as $y = -\sin\theta$.)

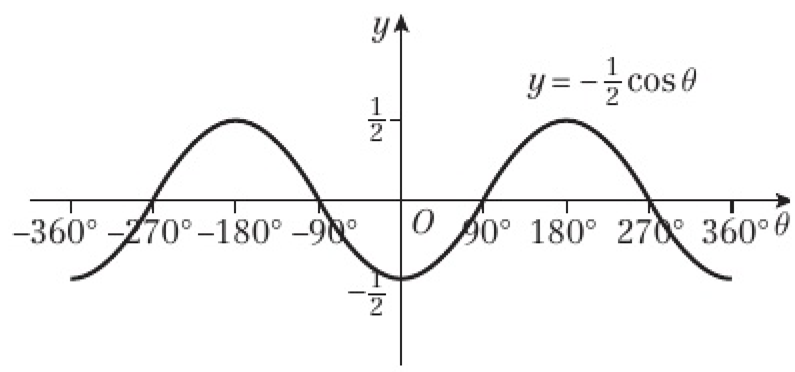


Meets θ-axis at (–180°, 0), (0°, 0), (180°, 0)
Maximum at (–90°, 1)
Minimum at (90°, –1)

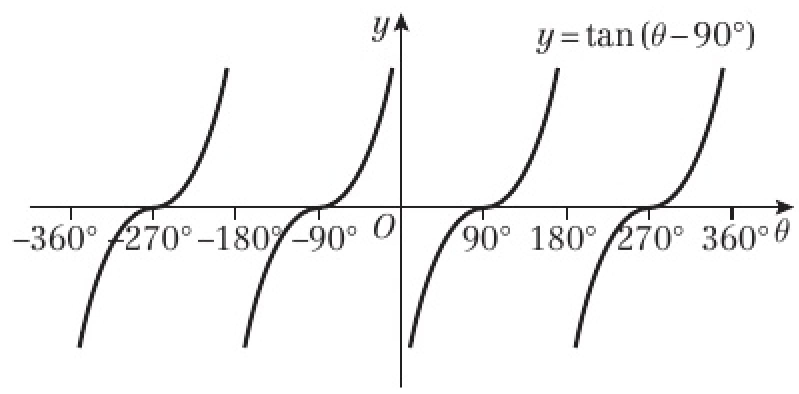
5 **a** Period = 720°



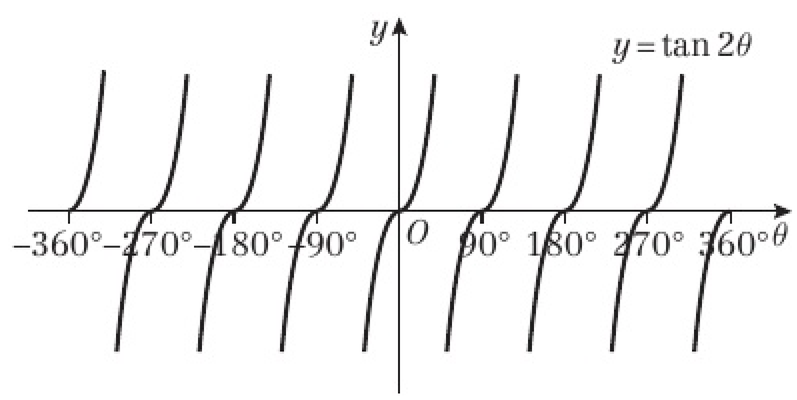
b Period = 360°



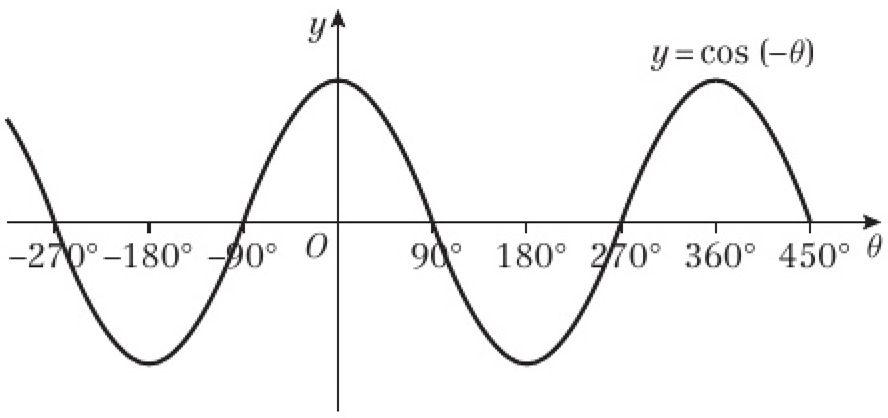
c Period = 180°



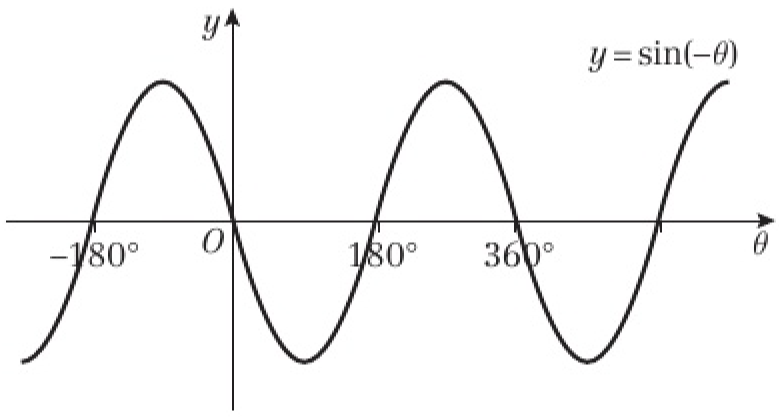
d Period = 90°



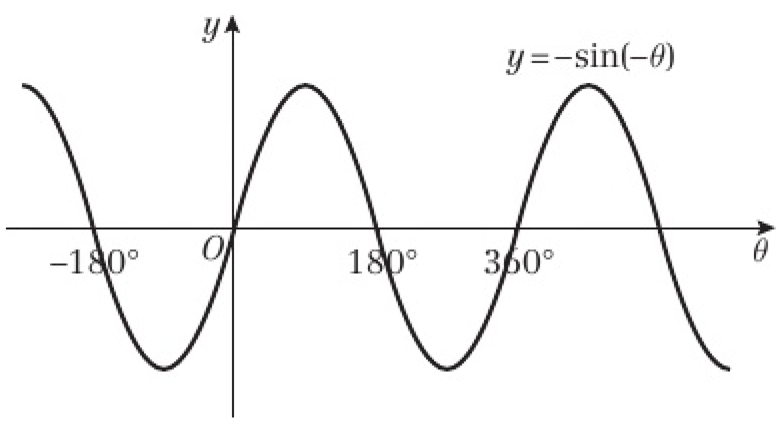
6 **a** **i** $y = \cos(-\theta)$ is a reflection of $y = \cos\theta$ in the y-axis, which is the same curve, so $\cos\theta = \cos(-\theta)$.



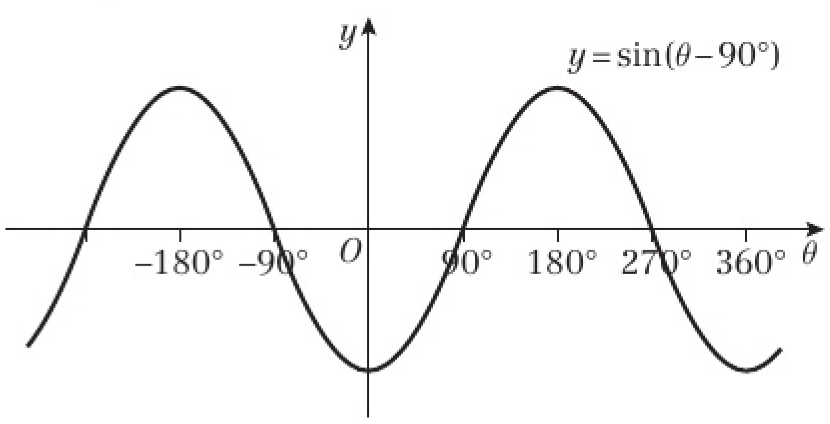
ii $y = \sin(-\theta)$ is a reflection of $y = \sin\theta$ in the y-axis.



$y = -\sin(-\theta)$ is a reflection of $y = \sin(-\theta)$ in the θ-axis, which is the graph of $y = \sin\theta$, so $-\sin(-\theta) = \sin\theta$.



iii $y = \sin(\theta - 90°)$ is the graph of $y = \sin\theta$ translated by 90° to the right, which is the graph of $y = -\cos\theta$, so $\sin(\theta - 90°) = -\cos\theta$.



b $\sin(90° - \theta)$
$= -\sin(-(90° - \theta)) = -\sin(\theta - 90°)$
using (a) (ii)
$= -(-\cos\theta)$ using (a) (iii)
$= \cos\theta$

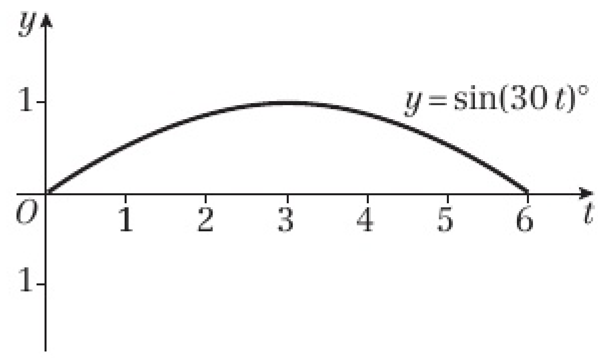
c Using (a)(i) $\cos(90° - \theta) = \cos(-(90° - \theta))$
$= \cos(\theta - 90°)$, but $\cos(\theta - 90°) = \sin\theta$,
so $\cos(90° - \theta) = \sin\theta$

7 **a** (−300°, 0), (−120°, 0), (60°, 0), (240°, 0)

b $\left(0°, \frac{\sqrt{3}}{2}\right)$

8 **a** $k = 60°$
b Yes – the graph of $y = \sin\theta$ repeats every 360°, so e.g. $k = 420°$.

9 **a**



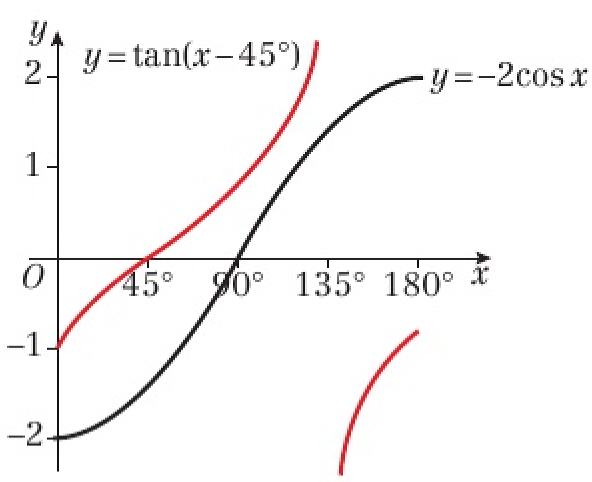
b Between 1 pm and 5 pm

Chapter review 6

1 **a** 155° **b** 13.7 cm
2 **a** $x = 49.5°$, area = 1.37 cm²
b $x = 55.2°$, area = 10.6 cm²
c $x = 117°$, area = 6.66 cm²
3 6.50 cm²
4 **a** 50.9 cm² **b** 12.0 cm²
5 **a** 5 **b** $\frac{25\sqrt{3}}{2}$ cm²
6 area $= \frac{1}{2}ab\sin C$
$1 = \frac{1}{2} \times 2\sqrt{2}\sin C$
$\frac{1}{\sqrt{2}} = \sin C \Rightarrow C = 45°$
Use the cosine rule to find the other side:
$x^2 = 2^2 + (\sqrt{2})^2 - 2 \times 2\sqrt{2}\cos C \Rightarrow x = \sqrt{2}$ cm
So the triangle is isosceles, with two 45° angles, thus is also right-angled.
7 **a** $AC = \sqrt{5}$, $AB = \sqrt{18}$, $BC = \sqrt{5}$

$$\cos\angle ACB = \frac{AC^2 + BC^2 - AB^2}{2 \times AC \times BC} = \frac{5 + 5 - 18}{2 \times \sqrt{5} \times \sqrt{5}} = -\frac{8}{10} = -\frac{4}{5}$$

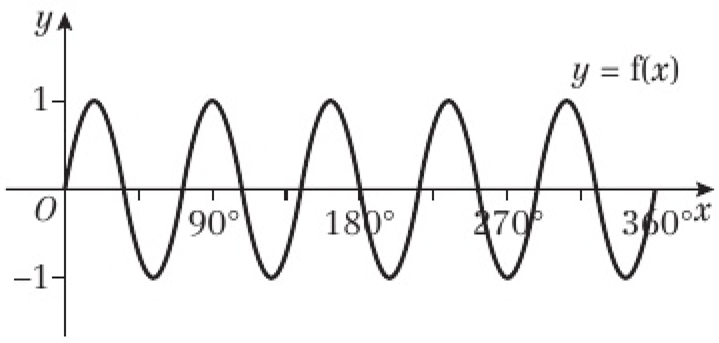
b $1\frac{1}{2}$ cm²
8 **a** 4 **b** $\frac{15\sqrt{3}}{4}$ (6.50) cm²
9 **a** 1.50 km **b** 241° **c** 0.789 km²
10 359 m²
11 35.2 m
12 **a** A stretch of scale factor 2 in the x direction.
b A translation of +3 in the y direction.
c A reflection in the x-axis.
d A translation of −20 in the x direction.
13 **a**



b There are no solutions.

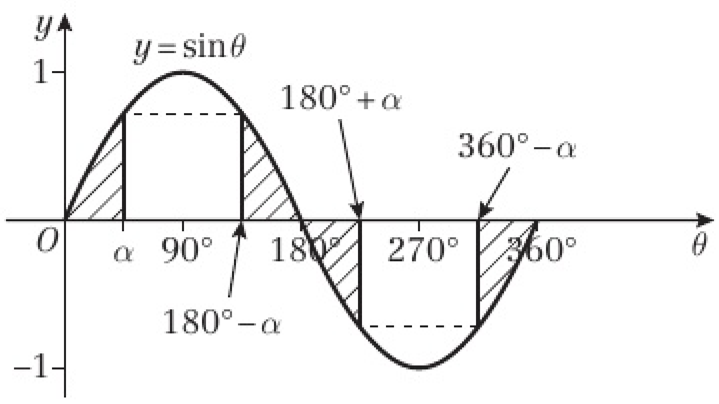
14 **a** 300° **b** (30°, 1) **c** 60° **d** $\frac{\sqrt{3}}{2}$

15 **a** $p = 5$



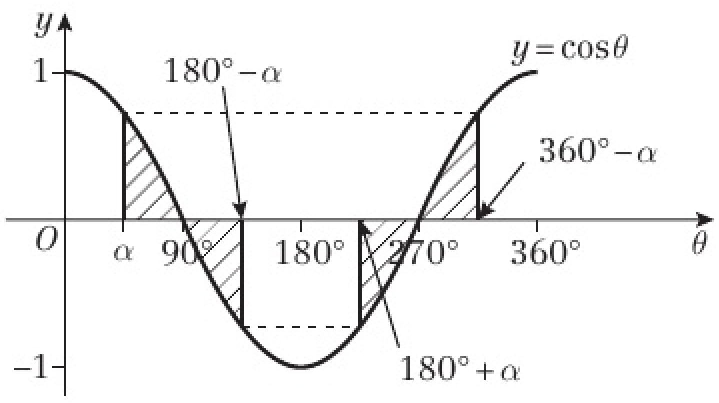
b 72°

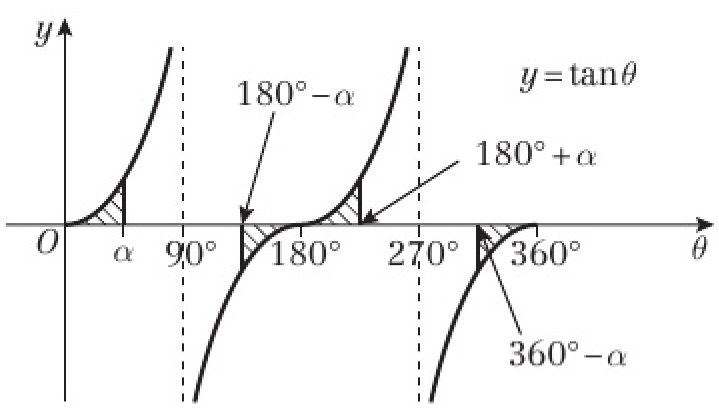
16 **a** The four shaded regions are congruent.



b $\sin\alpha$ and $\sin(180° - \alpha)$ have the same y value, (call it k)
so $\sin\alpha = \sin(180° - \alpha)$
$\sin(180° + \alpha)$ and $\sin(360° - \alpha)$ have the same y value, (which will be $-k$)
so $\sin\alpha = \sin(180° - \alpha)$
$= -\sin(180° + \alpha) = -\sin(360° - \alpha)$

17 **a**

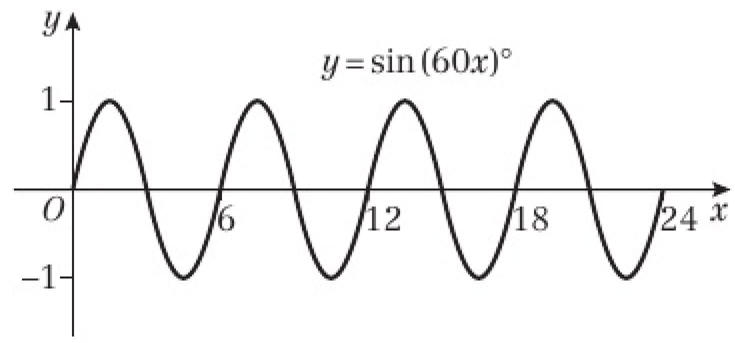




b **i** From the graph of $y = \cos\theta$, which shows four congruent shaded regions, if the y value at α is k, then y at $180° - \alpha$ is $-k$, y at $180° + \alpha = -k$ and y at $360° - \alpha = +k$
so $\cos\alpha = -\cos(180° - \alpha)$
$= -\cos(180° + \alpha) = \cos(360° - \alpha)$

ii From the graph of $y = \tan\theta$, if the y value at α is k, then at $180° - \alpha$ it is $-k$, at $180° + \alpha$ it is $+k$ and at $360° - \alpha$ it is $-k$,
so $\tan\alpha = -\tan(180° - \alpha)$
$= +\tan(180° + \alpha) = -\tan(360° - \alpha)$

18 **a**



b 4
c The dunes may not all be the same height.

Challenge
Using the sine rule:

$$\sin(180° - \angle ADB - \angle AEB) = \frac{5\left(\frac{1}{\sqrt{5}}\right)}{\sqrt{10}} = \frac{1}{\sqrt{2}}$$

$180° - \angle ADB - \angle AEB = 135°$ (obtuse)
so $\angle ADB + \angle AEB = 45° = \angle ACB$

CHAPTER 7

Prior knowledge 7

1 **a** $\frac{1}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** 0 **d** −1

2 $r = 1.59$ cm

3 length of side $AC = 30.4$ cm

4 area of triangle $ABC = 18\text{ cm}^2$

Exercise 7A

1 **a** 9° **b** 12° **c** 75° **d** 90°
e 140° **f** 210° **g** 225° **h** 270°
i 540°

2 **a** 26.4° **b** 57.3° **c** 65.0° **d** 99.2°
e 143.2° **f** 179.9° **g** 200.0°

3 **a** 0.479 **b** 0.156 **c** 1.74 **d** 0.909
e −0.897

4 **a** $\frac{2\pi}{45}$ **b** $\frac{\pi}{18}$ **c** $\frac{\pi}{8}$ **d** $\frac{\pi}{6}$
e $\frac{\pi}{4}$ **f** $\frac{\pi}{3}$ **g** $\frac{5\pi}{12}$ **h** $\frac{4\pi}{9}$
i $\frac{5\pi}{8}$ **j** $\frac{2\pi}{3}$ **k** $\frac{3\pi}{4}$ **l** $\frac{10\pi}{9}$
m $\frac{4\pi}{3}$ **n** $\frac{3\pi}{2}$ **o** $\frac{7\pi}{4}$ **p** $\frac{11\pi}{6}$

5 **a** 0.873 rad **b** 1.31 rad **c** 1.75 rad **d** 2.79 rad
e 4.01 rad **f** 5.59 rad

Exercise 7B

1 **a** **i** 2.7 cm **ii** 2.025 cm **iii** 7.5π cm
b **i** $\frac{50}{3}$ cm **ii** 1.8 cm **iii** 3.6 cm
c **i** $\frac{4}{3}$ **ii** 0.8 **iii** 2

2 $\frac{10}{3}\pi$ cm

3 2π cm

4 $5\sqrt{2}$

5 **a** 10.4 cm **b** 1.25 rad

6 7.5 cm

7 0.8

8 **a** $\frac{1}{3}\pi$ **b** $(6 + \frac{4}{3}\pi)$ cm

9 6.8 cm
10 a $R - r$

b $\sin\theta = \frac{r}{R-r} \Rightarrow (R-r)\sin\theta = r \Rightarrow (R\sin\theta - r\sin\theta) = r$

$\Rightarrow R\sin\theta = r + r\sin\theta \Rightarrow R\sin\theta = r(1+\sin\theta)$.

c 2.43 cm
11 2 rad
12 a 36 m b 13.6 km/h
13 a 3.5 m b 15.3 m
14 a 2.59 rad b 44 mm

Exercise 7C

1 a 19.2 cm^2 b $\frac{27}{4}\pi\text{ cm}^2$ c $\frac{162}{125}\pi\text{ cm}^2$

d 25.1 cm^2 e $(6\pi - 9\sqrt{3})\text{ cm}^2$ f $\left(\frac{63}{2}\pi + 9\sqrt{2}\right)\text{ cm}^2$

2 a $\frac{16}{3}\pi\text{ cm}^2$ b 5 cm^2
3 a 4.47 b 3.96 c 1.98
4 12 cm^2
5 a $\cos\theta = \frac{10^2 + 10^2 - 18.65^2}{2\times 10\times 10} = -0.739\ldots$

b 120 cm^2
6 $40\frac{2}{3}$ cm
7 a 12

b $A = \frac{1}{2}r^2\theta = \frac{1}{2}\times 12^2\times 0.5 = 36\text{ cm}^2$

c 1.48 cm^2

8 a $l = r\theta = \frac{x\pi}{12}$, $x = \frac{12l}{\pi}$

$A = \frac{1}{2}r^2\theta = \frac{1}{2}\left(\frac{12l}{\pi}\right)^2\frac{\pi}{12} = \frac{\pi}{24}\left(\frac{144l^2}{\pi^2}\right) = \frac{6l^2}{\pi}$

b 5π cm c 60

9 $\triangle COB = \frac{1}{2}r^2\sin\theta$

Shaded area $= \frac{1}{2}r^2(\pi - \theta) - \frac{1}{2}r^2\sin(\pi - \theta)$

$= \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta - \frac{1}{2}r^2(\sin\pi\cos\theta - \cos\pi\sin\theta)$

$= \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

Since $\triangle COB$ = shaded area,

$\frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2\pi - \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$

$\sin\theta = \pi - \theta - \sin\theta$

$\theta + 2\sin\theta = \pi$

10 38.7 cm^2
11 8.88 cm^2
12 a $OAD = \frac{1}{2}r^2\theta$, $OBC = \frac{1}{2}(r+8)^2\theta$

$ABCD = \frac{1}{2}(r+8)^2\theta - \frac{1}{2}r^2\theta = 48$

$\frac{1}{2}(r^2 + 16r + 64)\theta - \frac{1}{2}r^2\theta = 48$

$(r^2 + 16r + 64)\theta - r^2\theta = 96$

$16r + 64 = \frac{96}{\theta} \Rightarrow r = \frac{6}{\theta} - 4$

b 28 cm
13 78.4 ($\theta = 0.8$)
14 a $14^2 = 12^2 + 10^2 - 2\times 12\times 10\cos A$

$196 = 144 + 100 - 240\cos A$

$-48 = -240\cos A$

$0.2 = \cos A$

$A = \cos^{-1}(0.2) = 1.369438406\ldots = 1.37$ (3 s.f.)

b 34.1 m^2
15 a 18.1 cm

b 11.3 cm^2
16 a 98.79 cm^2

b 33.24 cm
17 4.62 cm^2

Challenge

Area $= \frac{1}{2}r^2\theta$, arc length, $l = r\theta$

Area $= \frac{1}{2}rl$

Chapter review 7

1 a $\frac{\pi}{3}$ b 8.56 cm^2
2 a 120 cm^2 b 161.07 cm^2
3 a 1.839 b 11.03 cm
4 a $\frac{p}{r}$ b Area $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\frac{p}{r} = \frac{1}{2}pr\text{ cm}^2$

c 12.207 cm^2 d $1.105 < \theta < 1.150$
5 a 1.28 b 16 c 1 : 3.91
6 a Area of shape $X = 2d^2 + \frac{1}{2}d^2\pi$

Area of shape $Y = \frac{1}{2}(2d)^2\theta$

$2d^2 + \frac{1}{2}d^2\pi = \frac{1}{2}(2d)^2\theta$

$2d^2 + \frac{1}{2}d^2\pi = 2d^2\theta \Rightarrow 1 + \frac{1}{4}\pi = \theta$

b $(3\pi + 12)$ cm c $\left(18 + \frac{3\pi}{2}\right)$ cm d 12.9 mm

7 a $A_1 = \frac{1}{2}\times 6^2\times\theta - \frac{1}{2}\times 6^2\times\sin\theta = 18(\theta - \sin\theta)$

b $A_2 = \pi\times 6^2 - 18(\theta - \sin\theta) = 36\pi - 18(\theta - \sin\theta)$

Since $A_2 = 3A_1$

$36\pi - 18(\theta - \sin\theta) = 3\times 18(\theta - \sin\theta)$

$36\pi - 18(\theta - \sin\theta) = 54(\theta - \sin\theta)$

$36\pi = 72(\theta - \sin\theta)$

$\frac{1}{2}\pi = \theta - \sin\theta$

$\sin\theta = \theta - \frac{\pi}{2}$

8 a $10^2 = 5^2 + 9^2 - 2\times 5\times 9\cos A$

$100 = 25 + 81 - 90\cos A$

$-6 = -90\cos A$

$\frac{1}{15} = \cos A$

$A = \cos^{-1}\left(\frac{1}{15}\right) = 1.504$

b i 6.77 cm^2 ii 15.7 cm^2 iii 22.5 cm

9 a $\frac{1}{2}r^2\times 1.5 = 15 \Rightarrow r^2 = 20$

$r = \sqrt{20} = 2\sqrt{5}$

b 15.7 cm c 5.025 cm^2
10 a $2\sqrt{3}$ cm b $2\pi\text{ cm}^2$

c Perimeter $= 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3}\times\frac{\pi}{3} = \frac{2\sqrt{3}}{3}(\pi + 6)$

11 a $70^2 = 44^2 + 44^2 - 2\times 44\times 44\cos C$

$\cos C = -\frac{257}{968}$

$C = \cos^{-1}\left(-\frac{257}{968}\right) = 1.84$

b i 80.9 m ii 26.7 m iii 847 m^2

12 a Arc $AB = 6\times 2\theta = 12\theta$

Length DC = Chord AB

$a^2 = 6^2 + 6^2 - 2\times 6\times 6\cos 2\theta = 72(1 - \cos 2\theta)$

$= 144\sin^2\theta$

$a = 12\sin\theta$

Perimeter $ABCD = 12\theta + 4 + 12\sin\theta + 4 = 2(7 + \pi)$

$12\theta + 12\sin\theta + 8 = 2(7 + \pi)$

$6\theta + 6\sin\theta - 3 = \pi$

$2\theta + 2\sin\theta - 1 = \frac{\pi}{3}$

b $2\times\frac{\pi}{6} + 2\sin\left(\frac{\pi}{6}\right) - 1 = \frac{\pi}{3} + 2\times\frac{1}{2} - 1 = \frac{\pi}{3}$

c 20.7 cm^2

13 a $O_1A = O_2A = 12$, as they are radii of their respective circles.
$O_1O_2 = 12$, as O_2 is on the circumference of C_1 and hence is a radius (and vice versa).
Therefore,
O_1AO_2 is an equilateral triangle $\Rightarrow \angle AO_1O_2 = \frac{\pi}{3}$.
By symmetry, $\angle BO_1O_2$ is $\frac{\pi}{3} \Rightarrow \angle AO_1B = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$
b 16π cm c 177 cm^2

14 a Student has used an angle measured in degrees – it needs to be measured in radians to use that formula.
b $\frac{5\pi}{4}$ cm^2

Challenge

a The angles subtended are 1.551 radians and 2.131 radians
b The area of region **R** is 8.9 cm^2.

CHAPTER 8

Prior knowledge check

1 a 5 b $-\frac{2}{3}$ c $\frac{1}{3}$
2 a x^{10} b $x^{\frac{2}{3}}$ c x^{-1} d $x^{\frac{3}{4}}$
3 a $y = \frac{1}{2}x - 2$ b $y = -\frac{1}{2}x + 8\frac{1}{2}$ c $y = -\frac{1}{4}x + 7\frac{1}{2}$
4 $y = -\frac{1}{2}x$

Exercise 8A

1 a

x-coordinate	−1	0	1	2	3
Estimate for gradient of curve	−4	−2	0	2	4

b Gradient = $2p - 2$ c 1

2 a $\sqrt{1 - 0.6^2} = \sqrt{0.64} = 0.8$
b Gradient = −0.75
c i −1.21 (3 s.f.) ii −1 iii −0.859 (3 s.f.)
d As other point moves closer to A, the gradient tends to −0.75.

3 a i 7 ii 6.5 iii 6.1
iv 6.01 v $h + 6$
b Gradient of tangent = 6

4 a i 9 ii 8.5 iii 8.1
iv 8.01 v $8 + h$
b Gradient of tangent = 8

Exercise 8B

1 a $f'(2) = \lim_{h\to 0}\frac{f(2+h) - f(2)}{h} = \lim_{h\to 0}\frac{(2+h)^2 - 2^2}{h}$
$= \lim_{h\to 0}\frac{4h + h^2}{h} = \lim_{h\to 0}(4 + h) = 4$

b $f'(-3) = \lim_{h\to 0}\frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0}\frac{(-3+h)^2 - 3^2}{h}$
$= \lim_{h\to 0}\frac{-6h + h^2}{h} = \lim_{h\to 0}(-6 + h) = -6$

c $f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{h^2 - 0^2}{h} = \lim_{h\to 0} h = 0$

d $f'(50) = \lim_{h\to 0}\frac{f(50+h) - f(50)}{h} = \lim_{h\to 0}\frac{(50+h)^2 - 50^2}{h}$
$= \lim_{h\to 0}\frac{100h + h^2}{h} = \lim_{h\to 0}(100 + h) = 100$

2 a $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$
$= \lim_{h\to 0}\frac{2xh + h^2}{h} = \lim_{h\to 0}(2x + h)$

b As $h \to 0$, $f'(x) = \lim_{h\to 0}(2x + h) = 2x$

3 a $g = \lim_{h\to 0}\frac{(-2+h)^3 - (-2)^3}{h}$
$= \lim_{h\to 0}\frac{-8 + 3(-2)^2h + 3(-2)h^2 + h^3 + 8}{h}$
$= \lim_{h\to 0}\frac{12h - 6h^2 + h^3}{h} = \lim_{h\to 0}(12 - 6h + h^2)$

b $g = 12$

4 a Gradient of $AB = \frac{(-1+h)^3 - 5(-1+h) - 4}{(-1+h) - (-1)}$
$= \frac{-1 + 3h - 3h^2 + h^3 + 5 - 5h - 4}{h}$
$= \frac{h^3 - 3h^2 - 2h}{h} = h^2 - 3h - 2$

b gradient = −2

5 $\frac{dy}{dx} = \lim_{h\to 0}\frac{6(x+h) - 6x}{h} = \lim_{h\to 0}\frac{6h}{h} = 6$

6 $\frac{dy}{dx} = \lim_{h\to 0}\frac{4(x+h)^2 - 4x^2}{h} = \lim_{h\to 0}\frac{8xh + 4h^2}{h}$
$= \lim_{h\to 0}(8x + 4h) = 8x$

7 $\frac{dy}{dx} = \lim_{h\to 0}\frac{a(x+h)^2 - ax^2}{h} = \lim_{h\to 0}\frac{(a-a)x^2 + 2axh + ah^2}{h}$
$= \lim_{h\to 0}\frac{2axh + ah^2}{h} = \lim_{h\to 0}(2ax + ah) = 2ax$

Challenge

a $f'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0}\frac{x - (x+h)}{xh(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)}$
$= \lim_{h\to 0}\frac{-1}{x^2 + xh}$

b $f'(x) = \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2 + xh} = \frac{-1}{x^2 + 0} = -\frac{1}{x^2}$

Exercise 8C

1 a $7x^6$ b $8x^7$ c $4x^3$ d $\frac{1}{3}x^{-\frac{2}{3}}$
e $\frac{1}{4}x^{-\frac{3}{4}}$ f $\frac{1}{3}x^{-\frac{2}{3}}$ g $-3x^{-4}$ h $-4x^{-5}$
i $-2x^{-3}$ j $-5x^{-6}$ k $-\frac{1}{2}x^{-\frac{3}{2}}$ l $-\frac{1}{3}x^{-\frac{4}{3}}$
m $9x^8$ n $5x^4$ o $3x^2$ p $-2x^{-3}$
q 1 r $3x^2$

2 a $6x$ b $54x^8$ c $2x^3$ d $5x^{-\frac{3}{4}}$
e $\frac{15}{2}x^{\frac{1}{4}}$ f $-10x^{-2}$ g $6x^2$ h $-\frac{1}{2x^5}$
i $x^{-\frac{3}{2}}$ j $\frac{15}{2}\sqrt{x}$

3 a $\frac{3}{4}$ b $\frac{1}{2}$ c 3 d 2

4 $\frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{2}}$

Exercise 8D

1 **a** $4x - 6$ **b** $x + 12$ **c** $8x$
d $16x + 7$ **e** $4 - 10x$

2 **a** 12 **b** 6 **c** 7
d $2\frac{1}{2}$ **e** -2 **f** 4

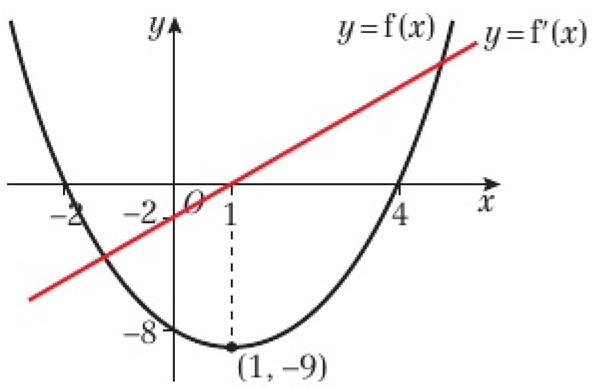
3 4, 0

4 $(-1, -8)$

5 1, -1

6 6, -4

7 **a, b**



c At the turning point, the gradient of $y = f(x)$ is zero, i.e. $f'(x) = 0$.

Exercise 8E

1 **a** $4x^3 - x^{-2}$ **b** $10x^4 - 6x^{-3}$ **c** $9x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

2 **a** 0 **b** $11\frac{1}{2}$

3 **a** $(2\frac{1}{2}, -6\frac{1}{4})$ **b** $(4, -4)$ and $(2, 0)$
c $(16, -31)$ **d** $(\frac{1}{2}, 4)$ $(-\frac{1}{2}, -4)$

4 **a** $x^{-\frac{1}{2}}$ **b** $-6x^{-3}$ **c** $-x^{-4}$
d $\frac{4}{3}x^3 - 2x^2$ **e** $\frac{1}{2}x^{-\frac{1}{2}} - 6x^{-4}$ **f** $\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-2}$
g $-3x^{-2}$ **h** $3 + 6x^{-2}$ **i** $5x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$
j $3x^2 - 2x + 2$ **k** $12x^3 + 18x^2$ **l** $24x - 8 + 2x^{-2}$

5 **a** 1 **b** $\frac{2}{9}$ **c** -4 **d** 4

6 $-\frac{3}{4}\sqrt{2}$

Exercise 8F

1 **a** $y + 3x - 6 = 0$ **b** $4y - 3x - 4 = 0$
c $3y - 2x - 18 = 0$ **d** $y = x$
e $y = 12x + 14$ **f** $y = 16x - 22$

2 **a** $7y + x - 48 = 0$ **b** $17y + 2x - 212 = 0$

3 $(1\frac{2}{9}, 1\frac{8}{9})$

4 $y = -x$, $4y + x - 9 = 0$; $(-3, 3)$

5 $y = -8x + 10$, $8y - x - 145 = 0$

6 $\left(-\frac{3}{4}, \frac{9}{8}\right)$

Challenge
L has equation $y = 12x - 8$.

Exercise 8G

1 **a** $24x + 3$, 24
b $15 - 3x^{-2}$, $6x^{-3}$
c $\frac{9}{2}x^{-\frac{1}{2}} + 6x^{-3}$, $-\frac{9}{4}x^{-\frac{3}{2}} - 18x^{-4}$
d $30x + 2$, 30
e $-3x^{-2} - 16x^{-3}$, $6x^{-3} + 48x^{-4}$

2 Acceleration $= \frac{3}{4}t^{-\frac{1}{2}} + \frac{3}{2}t^{-\frac{5}{2}}$

3 $\frac{3}{2}$

4 $-\frac{1}{2}$

Chapter review 8

1 $f'(x) = \lim_{h \to 0} \frac{10(x+h)^2 - 10x^2}{h} = \lim_{h \to 0} \frac{20xh + 10h^2}{h}$
$= \lim_{h \to 0} (20x + 10h) = 20x$

2 **a** y-coordinate of $B = (\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4$
Gradient $= \frac{((\delta x)^3 + 3(\delta x)^2 + 6\delta x + 4) - 4}{(1 + \delta x) - 1}$
$= \frac{(\delta x)^3 + 3(\delta x)^2 + 6\delta x}{(\delta x)} = (\delta x)^2 + 3\delta x + 6$
b 6

3 4, $11\frac{3}{4}$, $17\frac{25}{27}$

4 2, $2\frac{2}{3}$

5 $(2, -13)$ and $(-2, 15)$

6 **a** $1 - \frac{9}{x^2}$ **b** $x = \pm 3$

7 $\frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$

8 **a** $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$ **b** $(4, 16)$

9 **a** $x + x^{\frac{3}{2}} - x^{-\frac{1}{2}} - 1$ **b** $1 + \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$ **c** $4\frac{1}{16}$

10 $6x^2 + \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-2}$

11 $a = 1$, $b = -4$, $c = 5$

12 **a** $3x^2 - 10x + 5$
b **i** $\frac{1}{3}$ **ii** $y = 2x - 7$ **iii** $\frac{7}{2}\sqrt{5}$

13 $y = 9x - 4$ and $9y + x = 128$

14 **a** $\left(\frac{4}{5}, -\frac{2}{5}\right)$ **b** $\frac{1}{5}$

15 P is $(0, -1)$, $\frac{dy}{dx} = 3x^2 - 4x - 4$
Gradient at $P = -4$, so L is $y = -4x - 1$.
$-4x - 1 = x^3 - 2x^2 - 4x - 1 \Rightarrow x^2(x - 2) = 0$
$x = 2 \Rightarrow y = -9$, so Q is $(2, -9)$
Distance $PQ = \sqrt{(2 - 0)^2 + (-9 - (-1))^2} = \sqrt{68} = 2\sqrt{17}$

16 $(1, 4)$

17 **a** $\frac{250}{x^2} - 2x$ **b** $(5, 125)$

18 **a** $P\left(x, 5 - \frac{1}{2}x^2\right)$
$OP^2 = (x - 0)^2 + \left(5 - \frac{1}{2}x^2 - 0\right)^2$
$= \frac{1}{4}x^4 - 4x^2 + 25$
b $x = \pm 2\sqrt{2}$ or $x = 0$
c When $x = \pm 2\sqrt{2}$, $f''(x)$ so minimum
When $x = \pm 2\sqrt{2}$, $y = 9$ so $OP = 3$

Challenge
When $x = 2$, $y = -7$

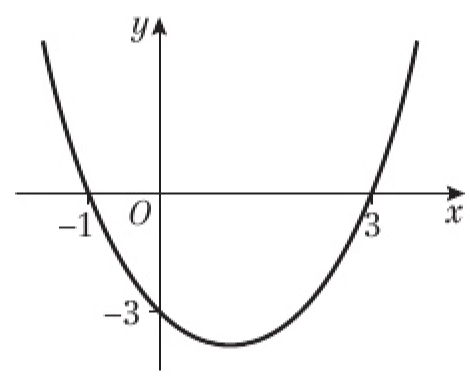
CHAPTER 9

Prior knowledge check

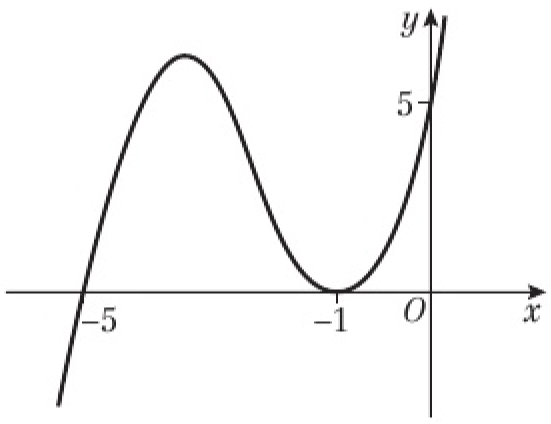
1 **a** $x^{\frac{5}{2}}$ **b** $2x^{\frac{3}{2}}$ **c** $x^{\frac{5}{2}} - \sqrt{x}$ **d** $x^{-\frac{3}{2}} + 4x$

2 **a** $6x^2 + 3$ **b** $x - 1$ **c** $3x^2 + 2x$ **d** $-\frac{1}{x^2} - 3x^2$

3 **a**



b



Exercise 9A

1 a $y = \frac{1}{6}x^6 + c$ b $y = 2x^5 + c$
c $y = x^{-1} + c$ d $y = 2x^{-2} + c$
e $y = \frac{3}{5}x^{\frac{5}{3}} + c$ f $y = \frac{8}{3}x^{\frac{3}{2}} + c$
g $y = -\frac{2}{7}x^7 + c$ h $y = 2x^{\frac{1}{2}} + c$
i $y = -10x^{-\frac{1}{2}} + c$ j $y = \frac{9}{2}x^{\frac{4}{3}} + c$
k $y = 3x^{12} + c$ l $y = 2x^{-7} + c$
m $y = -9x^{\frac{1}{3}} + c$ n $y = -5x + c$
o $y = 3x^2 + c$ p $y = \frac{10}{3}x^{0.6} + c$

2 a $y = \frac{1}{4}x^4 - 3x^{\frac{1}{2}} + 6x^{-1} + c$ b $y = x^4 + 3x^{\frac{1}{3}} + x^{-1} + c$
c $y = 4x + 4x^{-3} + 4x^{\frac{1}{2}} + c$ d $y = 3x^{\frac{5}{3}} - 2x^5 - \frac{1}{2}x^{-2} + c$
e $y = 4x^{-\frac{1}{3}} - 3x + 4x^2 + c$ f $y = x^5 + 2x^{-\frac{1}{2}} + 3x^{-4} + c$

3 a $f(x) = 6x^2 - 3x^{-\frac{1}{2}} + 5x + c$ b $f(x) = x^6 - x^{-6} + x^{-\frac{1}{6}} + c$
c $f(x) = x^{\frac{1}{2}} + x^{-\frac{1}{2}} + c$ d $f(x) = 2x^5 - 4x^{-2} + c$
e $f(x) = 3x^{\frac{2}{3}} - 6x^{-\frac{2}{3}} + c$
f $f(x) = 3x^3 - 2x^{-2} + \frac{1}{2}x^{\frac{1}{2}} + c$

4 $y = \dfrac{4x^3}{3} + 6x^2 + 9x + c$

5 $f(x) = -3x^{-1} + 4x^{\frac{3}{2}} + \dfrac{x^2}{2} - 4x + c$

Challenge

$y = -\dfrac{12}{7x^{\frac{7}{2}}} - \dfrac{4}{5x^{\frac{5}{2}}} + \dfrac{3}{2x^2} + \dfrac{1}{x} + c$

Exercise 9B

1 a $\dfrac{x^4}{4} + c$ b $\dfrac{x^8}{8} + c$
c $-x^{-3} + c$ d $\dfrac{5x^3}{3} + c$

2 a $\frac{1}{5}x^5 + \frac{1}{2}x^4 + c$ b $\dfrac{x^4}{2} - \dfrac{x^3}{3} + \dfrac{5x^2}{2} + c$
c $2x^{\frac{5}{2}} - x^3 + c$

3 a $-4x^{-1} + 6x^{\frac{1}{2}} + c$ b $-6x^{-1} - \frac{2}{3}x^{\frac{3}{2}} + c$
c $-4x^{-\frac{1}{2}} + \dfrac{x^3}{3} - 2x^{\frac{1}{2}} + c$

4 a $x^4 + x^{-3} + 2x + c$ b $\frac{1}{2}x^2 + 2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + c$
c $\dfrac{px^5}{5} + 2qx - 3x^{-1} + c$

5 a $t^3 + t^{-1} + c$ b $\frac{2}{3}t^3 + 6t^{-\frac{1}{2}} + t + c$
c $\dfrac{p}{4}t^4 + q^2t + pr^3t + c$

6 a $x^2 - \dfrac{3}{x} + c$ b $\frac{4}{3}x^3 + 6x^2 + 9x + c$
c $\frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + c$

7 a $\frac{1}{3}x^3 + 2x - \dfrac{1}{x} + c$ b $\frac{1}{2}x^2 + \frac{8}{3}x^{\frac{3}{2}} + 4x + c$
c $2x^{\frac{1}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$

8 a $\frac{3}{5}x^{\frac{5}{3}} - \dfrac{2}{x^2} + c$ b $-\frac{1}{x^2} - \frac{1}{x} + 3x + c$
c $\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 - 3x + c$ d $\frac{8}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$
e $3x + 2x^{\frac{1}{2}} + 2x^3 + c$ f $\frac{2}{5}x^{\frac{5}{2}} + 3x^2 + 6x^{\frac{3}{2}} + c$

9 a $-\dfrac{A}{x} - 3x + c$ b $\frac{2}{3}\sqrt{P}\,x^{\frac{3}{2}} - \dfrac{1}{x^2} + c$
c $-\dfrac{p}{x} + \dfrac{2qx^{\frac{3}{2}}}{3} + rx + c$

10 $-\dfrac{6}{x} + \dfrac{8x^{\frac{3}{2}}}{3} - \dfrac{3x^2}{2} + 2x + c$

11 $2x^4 + 3x^2 - 6x^{\frac{1}{2}} + c$

12 a $(2 + 5\sqrt{x})^2 = 4 + 10\sqrt{x} + 10\sqrt{x} + 25x = 4 + 20\sqrt{x} + 25x$
b $4x + \dfrac{40x^{\frac{3}{2}}}{3} + \dfrac{25x^2}{2} + c$

13 $\dfrac{x^6}{2} - 8x^{\frac{1}{2}} + c$

14 $p = -4, q = -2.5$

Exercise 9C

1 a $y = x^3 + x^2 - 2$ b $y = x^4 - \dfrac{1}{x^2} + 3x + 1$
c $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{12}x^3 + \frac{1}{3}$ d $y = 6\sqrt{x} - \frac{1}{2}x^2 - 4$
e $y = \frac{1}{3}x^3 + 2x^2 + 4x + \frac{2}{3}$ f $y = \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + 1$

2 $f(x) = \frac{1}{2}x^4 + \frac{1}{x} + \frac{1}{2}$

3 $y = 1 - \dfrac{2}{\sqrt{x}} - \dfrac{3}{x}$

4 $f(x) = 3x^3 + 2x^2 - 3x - 2$

5 $y = 6x^{\frac{1}{2}} - \dfrac{4x^{\frac{5}{2}}}{5} + \dfrac{118}{5}$

6 a $p = \frac{1}{2}, q = 1$ b $y = 4x^{\frac{3}{2}} + \dfrac{5x^2}{2} - \frac{421}{2}$

7 a $f(t) = 10t - \dfrac{5t^2}{2}$ b $7\frac{1}{2}$

8 a $f(t) = -4.9t^2 + 35$ b 23.975 m
c 35 m d 2.67 seconds
e e.g. the ground is flat

Challenge

1 $f_2(x) = \dfrac{x^3}{3}$; $f_3(x) = \dfrac{x^4}{12}$ b $\dfrac{x^{n+1}}{3 \times 4 \times 5 \times \ldots \times (n+1)}$

2 $f_2(x) = x + 1$; $f_3(x) = \frac{1}{2}x^2 + x + 1$; $f_4(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$

Chapter review 9

1 a $\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x + c$ b $\frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + c$

2 $\frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + \frac{1}{6}$

3 a $2x^4 - 2x^3 + 5x + c$ b $2x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$

4 $\frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c$

5 $x = \frac{1}{3}t^3 + t^2 + t - 8\frac{2}{3}$; $x = 12\frac{1}{3}$

6 a $A = 6, B = 9$
b $\frac{3}{5}x^{\frac{5}{3}} + \frac{9}{2}x^{\frac{4}{3}} + 9x + c$

7 a $\frac{9}{2}x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}$
b $6x^{\frac{3}{2}} + 32x^{\frac{1}{2}} - 24x + c$

8 $a = 4, b = -3.5$

9 25.9 m

10 a $f(t) = 5t + t^2$ b 7.8 seconds

11 **a** $\frac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$ **b** $2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$

12 **a** $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$

b (4,16)

13 $-\frac{9}{x} - \frac{16x^{\frac{3}{2}}}{3} + 2x^2 - 5x + c$

14 **a** $f'(x) = \frac{(2 - x^2)(4 - 4x^2 + x^4)}{x^2} = 8x^{-2} - 12 + 6x^2 - x^4$

b $f(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$

Challenge

a $k = 3$

b The equation of the curve is $y = 2x^3 - 3x^2 + 3x + 2$

Review exercise 2

1 $x + 3y - 22 = 0$

2 $x - 3y - 21 = 0$

3 4, –2.5

4 **a** $y = -\frac{1}{3}x + 4$

b C is (3, 3)

c 15

5 $\sqrt{10}$ cm

6 **a** $\cos 60° = \frac{1}{2} = (5^2 + (2x - 3)^2 - (x + 1)^2) \div 2(5)(2x - 3)$

$5(2x - 3) = (25 + 4x^2 - 12x + 9 - x^2 - 2x - 1)$

$0 = 3x^2 - 24x + 48$

$x^2 - 8x + 16 = 0$

b 4

c 10.8 cm²

7 **a** 11.93 km

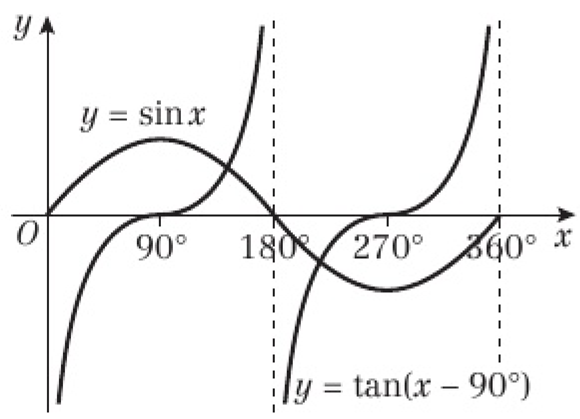
b 100.9°

8 **a** $AB = BC = 10$ cm, $AC = 6\sqrt{10}$ cm

b 143.1°

9 19.4 cm²

10 **a**



b 2

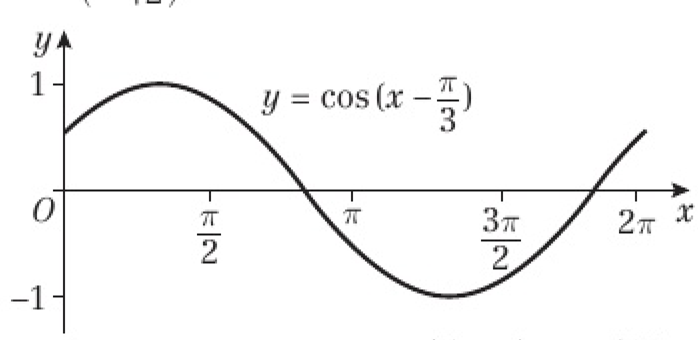
11 **a** (–225, 0), (–45, 0), (135, 0) and (315, 0)

b $\left(0, \frac{\sqrt{2}}{2}\right)$

12 x-axis: $\left(-\frac{7\pi}{4}, 0\right), \left(-\frac{3\pi}{4}, 0\right), \left(\frac{\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right)$

y-axis: $\left(0, \frac{1}{\sqrt{2}}\right)$

13 **a**



b y-axis at (0, 0.5). x-axis at $\left(\frac{5\pi}{6}, 0\right)$ and $\left(\frac{11\pi}{6}, 0\right)$

c $x = 2.89$, $x = 5.49$

14 **a** 1.287 radians **b** 6.44 cm

15 $(12 + 2\pi)$ cm

16 **a** $\frac{1}{2}(r + 10)^2\theta - \frac{1}{2}r^2\theta = 40 \Rightarrow 20r\theta + 100\theta = 80$

$\Rightarrow r\theta + 5\theta = 4 \Rightarrow r = \frac{4}{\theta} - 5$

b 28 cm

17 **a** 6 cm **b** 6.7 cm²

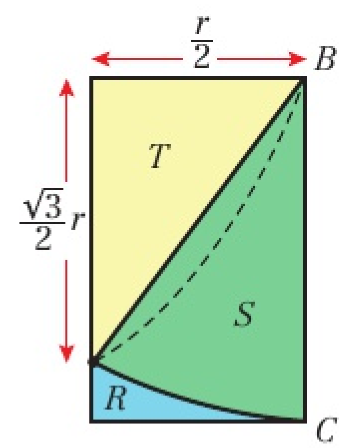
18 **a** 119.7 cm² **b** 40.3 cm

19 Split each half of the rectangle as shown.

Area $S = \frac{\pi}{12}r^2$

Area $T = \frac{\sqrt{3}}{8}r^2$

$\Rightarrow$ Area $R = \left(\frac{1}{2} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)r^2$



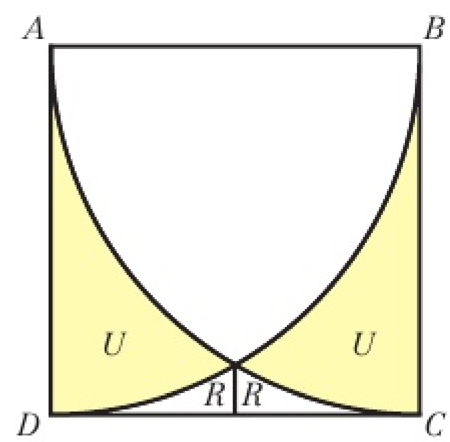
$U = \left(r^2 - \frac{\pi}{4}r^2\right) - 2R$

$= \left(1 - \frac{\pi}{4} - 1 + \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)r^2$

$= r^2\left(\frac{\sqrt{3}}{4} - \frac{\pi}{12}\right)$

$\therefore$ Shaded area

$= \frac{r^2}{12}(3\sqrt{3} - \pi)$



20 $\lim_{h \to 0} \frac{5(x + h)^2 - 5x^2}{h}$

$= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 5x^2}{h}$

$= \lim_{h \to 0} \frac{10xh + 5h^2}{h}$

$= \lim_{h \to 0} 10x + 5h$

$= 10x$

21 $\frac{dy}{dx} = 12x^2 + x^{-\frac{1}{2}}$

22 **a** $\frac{dy}{dx} = 4 + \frac{9}{2}x^{\frac{1}{2}} - 4x$

b Substitute $x = 4$ into equation for C

c Gradient of tangent = –3 so gradient of normal = $\frac{1}{3}$

Substitute (4, 8) into $y = \frac{1}{3}x + c$

Rearrange $y = \frac{1}{3}x + \frac{20}{3}$

d $PQ = 8\sqrt{10}$

23 **a** $\frac{dy}{dx} = 8x - 5x^{-2}$, at P this is 3

b $y = 3x + 5$

c $k = -\frac{5}{3}$

24 **a** $P = 2, Q = 9, R = 4$ **b** $3x^{\frac{1}{2}} + \frac{9}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$

c When $x = 1$, $f'(x) = 5\frac{1}{2}$, gradient of $2y = 11x + 3$ is $5\frac{1}{2}$, so it is parallel with tangent

25 **a** $\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$ **b** $\frac{d^2y}{dx^2} = 6 - x^{-\frac{3}{2}}$

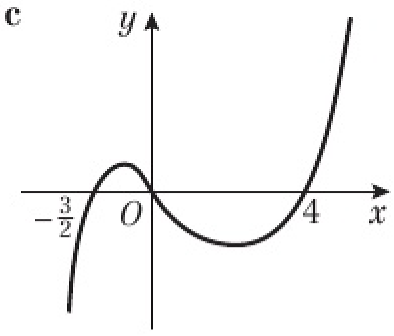
c $x^3 + \frac{8}{3}x^{\frac{3}{2}} + c$

Online Worked solutions are available in SolutionBank.

26 a $2x^3 - 5x^2 - 12x$

b $2x^3 - 5x^2 - 12x$
$= x(2x^2 - 5x - 12)$
$= x(2x + 3)(x - 4)$

c



Challenge

1 a 160

b $\left(-\frac{28}{3}, 0\right)$

2 $\frac{\pi - 2}{2 + 3\pi} : 1$

3 a $f'(-3) = f'(2) = 0$, so $f'(x) = k(x + 3)(x - 2)$
$= k(x2 + x - 6)$;
there are no other factors as $f(x)$ is cubic.

b $2x^3 + 3x^2 - 36x - 5$

Exam Practice

1 a $28a^2b$ **b** $48a^6b$ **c** $-2 - \sqrt{7}$

2 $x(4x - 3)(x + 1)$

3 $\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - 3$ or $\frac{2}{3}x^{1\frac{1}{2}} - 2x^{\frac{1}{2}} - 3$ or $\frac{2}{3}x^{1.5} - 2x^{0.5} - 3$

4 a $A = \frac{1}{5}$ or $A = 0.2$, $p = 1.5$ or $p = 1\frac{1}{2}$, $B = 2$, $q = -0.5$ or $q = -\frac{1}{2}$

b $0.3x^{0.5} - x^{-1.5}$

c $0.08x^{2.5} + 4x^{-0.5}$

5 a max (–4, 18) min (2, –6)

b max (–4, 0) min (2, –12)

c max (–2, 9) min (0, –3)

d max (–2, 9) min (1, –3)

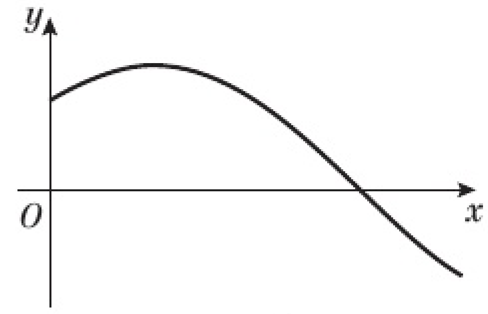
6 $4x + 3x^{-\frac{2}{3}} + \frac{1}{4}x^{2\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}$ or $4x + 3x^{-\frac{2}{3}} + \frac{1}{4}x^{2.5} - \frac{3}{2}x^{-0.5}$

7 a Different real roots, discriminant > 0
so $k^2 - 4(k + 4) \times 1 = k^2 - 4k - 16 > 0$

b $k < 2 - 2\sqrt{5}$ or $k > 2 + 2\sqrt{5}$

8 $p = 2$, $q = -3$

9 a



b x-axis crossing: $\frac{1}{\sqrt{2}}, 0$

y-axis crossing: $1 + \frac{\pi}{4}, 0$

10 (3, 4)

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